DIGITAL TOPOLOGY WITH APPLICATIONS TO IMAGE PROCESSING

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ABSTRACT

Digital topology is the study of the topological properties of machine-generated binary image arrays. The underlying theme of this thesis is the construction of natural continuous analogues for discrete binary images, which clarify the geometric meaning of digital pictures, and make it possible to apply concepts and theorems of polyhedral topology to them. This leads to substantial extensions and generalizations of the work of earlier authors.

As is well-known, one proves that a proposed two-dimensional parallel thinning algorithm is topologically sound by establishing that each component of the input black set contains exactly one component of the skeleton and that each component of the complement of the skeleton contains exactly one component of the input white set. But this method does not suffice for three-dimensional thinning algorithms. Consequently, there is a need for a precise definition of what it means for a three-dimensional thinning algorithm to "preserve topology". In the present thesis the concept of a continuous analogue is used to give a definition which is both simple and natural. The relationship of our approach to that of others who have worked on this difficult problem is carefully investigated.

The theory of digital pictures presented here differs from the conventional theory in that it admits a very much wider range of adjacency relations. As a consequence of this, the new theory turns out to be a suitable basis for establishing certain fundamental connectivity and separation properties of digital borders, which are of much significance to the theory of border-tracking algorithms. Essential use is made of continuous analogues in this work.

In another direction, a discrete three-dimensional Jordan-Brouwer Separation Theorem (the analogous result in three-space of the celebrated Jordan Curve Theorem) is proved by means of continuous analogues. (Some special cases of this result were established by direct --- and quite difficult --- combinatorial arguments in a series of recent papers by Morgenthaler, Reed and Rosenfeld.) The method of continuous analogues also provides a visual interpretation of one of the obstacles encountered in constructing a direct proof, namely the "cross-cap".
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Chapter 1 - INTRODUCTION

Digital Topology

Digital topology is the study of the topological properties of machine-generated binary image arrays. The subject has important uses in the theory of image-thinning, a common pre-processing technique in automated pattern recognition. Other applications include the development and proof of soundness of border-tracking algorithms. This thesis covers most areas of digital topology.

The underlying theme of our work is to relate digital topology to classical topology, in the main part by constructing natural polyhedral analogues for discrete binary images. Thus a correspondence between discrete images and "continuous" sets is established, which not only increases our understanding of the geometric meaning of digital pictures, but also allows us to apply concepts and theorems of polyhedral topology to them. This leads to substantial extensions and generalizations of the work of earlier authors.

The only image arrays considered in this thesis are "threeshinner" arrays, all of whose elements have value 0 or 1. For accounts of fuzzy digital topology, which aims to generalize digital topology to unsegmented "grey-scale" image arrays (whose elements lie in the range $0 \leq x \leq 1$) see [Rosenfeld79] and [Rosenfeld84]. Additionally, we confine our attention to two- and three-dimensional image arrays. Higher dimensional arrays have been considered by some authors (e.g. [TourlakisMylopoulos73, UdupaSrihariHerman79]).

A Brief Outline of the Thesis

In digital topology it is quite common to model a digital picture as a structure consisting of black and white lattice-points, together with a definition of when two points are adjacent to each other. Of course, the black lattice-points correspond to the 1's in the binary image array. We too use this approach, but we place far fewer restrictions than
usual on the adjacency relations employed. This extra generality produces two tangible benefits.

First of all, much of our work can be applied to the two-dimensional hexagonal and three-dimensional face-centred cubic and body-centred cubic lattices by the simple device of using an affine map to transform them to the familiar rectangular lattice. The point is that isotropic adjacency relations on the original lattice will be mapped onto anisotropic relations on the rectangular lattice, and our theory of digital pictures does admit these.

The second advantage of our theory is that it allows alteration of the sets of black and white points without changing the underlying adjacency relation, and vice versa. In the conventional theory, it can happen that when two adjacent black points are changed into white points (or vice versa) then the points are no longer adjacent. In our theory such changes in the adjacency relation need not happen if they are inconvenient. Conversely, we can if we wish change the adjacency relation between two points without changing their colours. The extra flexibility gained from our theory can be extremely useful, as can be seen from the proofs of Propositions 4 and 5 in Chapter 2.

The notion of a continuous analogue of a digital picture is defined in Chapter 2, which is a modified version of [KrmgRosen85c]. Continuous analogues serve three purposes. First, they can be used to translate topological problems concerning digital pictures into problems of polyhedral topology, which can often then be solved by application of standard techniques or well-known results. This is illustrated in Chapter 2, where continuous analogues are proved to exist for all suitably well behaved digital pictures, and are used to extend the theorem of [Rosenfeld74] -- which states that the adjacency graph of any two-dimensional digital picture based on the (4,8) or (8,4) adjacency relations is a tree -- to a very large class of three-dimensional adjacency relations. This result enables us to establish certain simple but deep connectivity and separation properties of digital borders, which have been often been assumed without proof in work on border-tracking algorithms.
Chapters 3 and 4 of the thesis present a general theory of two- and three-dimensional image shrinking algorithms. Here we come to the second purpose of continuous analogues, which is to provide a systematic way of applying concepts of polyhedral topology to digital pictures. Thus we define the Euler characteristic of a digital picture to be the Euler characteristic of a certain standard continuous analogue of the picture, and we say that one digital picture is a shrunk image of another if the standard continuous analogue of the first collapses to the standard continuous analogue of the second. (A polyhedron \( P \) is said to collapse to a polyhedron \( Q \subseteq P \) if there is a continuous deformation of a very simple kind -- see Chapter 4 -- which transforms \( P \) into \( Q \) in such a way that the points in \( Q \) remain fixed throughout the process.) As might be surmised from its title "Standard Continuous Analogues and the Euler Characteristic", Chapter 3 develops basic theory and will hopefully be of independent interest apart from its relevance to image shrinking. The significance of our work to the theory of image thinning will be discussed at greater length later on in this chapter.

In Chapter 5 continuous analogues are again used as tools for transforming digital topology to polyhedral topology. We give a relatively straightforward proof of a discrete three-dimensional Jordan-Brouwer Separation Theorem (the analogue for closed surfaces in three-space of the celebrated Jordan Curve Theorem). Special cases of this were proved by direct (and quite difficult) combinatorial arguments in a series of recent papers by Morgenthaler, Reed and Rosenfeld ([MorgenthalerRosenfeld81], [ReedRosenfeld84], [Reed84]). Chapter 5 also provides a good example of the third use of continuous analogues, which is to provide simple visual interpretations of complex discrete structures. In our case the "complex discrete structures" are the neighbourhoods of the simple surface points defined by Morgenthaler and Rosenfeld. The visual interpretation enables us to understand the "cross-cap", which was one of the major difficulties encountered by the earlier authors in their direct attack on the problem. Chapter 5 is a modified version of [KongRoscoe85a].
Terminology

Before proceeding further we define some basic terminology, which is essentially that used by Rosenfeld (Rosenfeld70, Rosenfeld81). The lattice-points in three-space are said to be 26-adjacent if they are distinct and each coordinate of one differs from the corresponding coordinate of the other by at most 1; two points are 18-adjacent if they are 26-adjacent and differ in at most two of their coordinates; two points are 6-adjacent if they are 26-adjacent and differ in at most one coordinate. Two lattice-points in the plane \((x, y)\) and \((x', y')\) are said to be 8-adjacent or 4-adjacent according as \((x, y, 0)\) and \((x', y', 0)\) are 26-adjacent or 6-adjacent. For \(a = 4, 8, 6, 18, 26\) we say a set \(S\) of lattice points is \(a\)-connected if \(S\) cannot be partitioned into two subsets which are not \(a\)-adjacent to each other. An \(a\)-component of \(S\) is a set of lattice points \(S\) which is not \(a\)-adjacent to any other point in \(S\). A simple closed \(a\)-curve is an \(a\)-connected set of lattice-points each of which is \(a\)-adjacent to exactly two other points in the set. An \((a, b)\) digital picture is one in which two black points are considered to be adjacent if they are \(a\)-adjacent, and two white points or a white point and a black point are considered to be adjacent if they are \(b\)-adjacent.

Image Shrinking and Thinning

A thinning algorithm works by selectively deleting ("whitening") black points, to produce a 'skeleton' of the input black set. (Some algorithms operate in a rather more complex way.) The output black set should everywhere be at most two pixels/voxels thick -- ideally only one pixel/voxel thick -- and each elongated part of the input black set should be represented by a black "digital arc" in the output. Most importantly, the thinning process should 'preserve the topology' of the input black set: this is why digital topology is relevant.

The topological considerations involved in the design of two-dimensional thinning algorithms
have been sufficiently well understood for some years, and certainly since the appearance of Stefanelli and Rosenfeld's paper [StefanelliRosenfeld71]. In the case of (8,4) digital pictures, Hilditch gave a simple and satisfactory definition of topology preservation in [Hilditch69]. Like many other authors (and in contrast to our approach), Hilditch identified each black point p with the unit square whose sides are parallel to the coordinate axes and whose centre is at p. She stated that a thinning algorithm was topology preserving if and only if the output skeleton (as a union of unit squares) could be obtained by a continuous deformation of the input black set.

In their paper, Stefanelli and Rosenfeld used another criterion to establish the soundness of one of their parallel thinning algorithms: a two-dimensional thinning algorithm preserves topology if and only if each black component of the input contains exactly one component of the skeleton, and each component of the complement of the skeleton contains exactly one white component of the input image. As we are not aware of any published proof of the equivalence of this widely used practical criterion to the more natural definition given by Hilditch, we give our own proof in Chapter 4 (Theorem 1').

Two-dimensional thinning remains an area of active research, but the efforts of researchers are now focused upon the non-topological aspects of algorithm-performance. The ultimate goal of the work may be briefly described as that of constructing efficient algorithms which satisfy the conditions we have already stated, and have the following additional properties: all significant protrusions of the input image produce arcs in the skeleton, but the skeleton is free of "spurs" caused by minor surface irregularities in the input image; the skeleton is insensitive to input "noise" in the form of small components of 1's and 0's; each arc in the skeleton is symmetrically placed with respect to the portion of the input image that it represents; the shape of the skeleton is invariant under rotations, reflections and enlargements of the input image. Algorithms for generation of labelled skeletons, from which the original image can be reconstructed (either exactly or approximately) are also of interest. The problems still to be solved are difficult, and beyond the scope of this thesis.
For general surveys of the literature on thinning of two-dimensional binary images (and further references) see [Tamura78], [DaviesPlummer81] and [NaccacheShingal84]. Interesting recent papers not mentioned in these surveys are [Arcelli81] and [Hilditch83]. Algorithms have been proposed for thinning grey-scale -- as opposed to binary -- images (e.g. [Hilditch69], [DyerRosenfeld79], [SaliariSiv82a,82b] and [Hilditch83]).

Although the topological problems of two-dimensional thinning have mostly been solved, the position with regard to three-dimensional thinning is different. It is convenient to use the term "shrinking" for the simpler process of reducing the number of black points in an image without affecting the topology of that image. The difference between shrinking and thinning can be seen from the fact that an arc should be invariant under thinning, but can be shrunk to a single point. (On the other hand closed curves are invariant under shrinking.) Being free of the tricky non-lupological requirements we mentioned above, shrinking is much easier than thinning. But in three dimensions even shrinking poses some challenging problems, since the concept of topology preservation is much subtler in three-space than in the plane. Note that shrinking in our sense is different from the notion of "shrinking" used in [Levialdi72] and [KaoPrasadarSarma76], in which "holes" are not preserved. Our usage is that of [Rosenfeld73] and [Pavlidis80].

As we mentioned, one proves that a proposed two-dimensional parallel thinning (or shrinking) algorithm is topologically sound by establishing that each component of the input black set contains exactly one component of the skeleton and that each component of the complement of the skeleton contains exactly one component of the input white set. Unfortunately, this simple criterion is totally inadequate in three dimensions. In fact, although several authors have stated conditions which should be satisfied by a black point if the topology of the picture is not to change when that single point is deleted, there has so far been no universally accepted definition of "preserves topology" when many black points are deleted at the same time. It is plain that such a definition is essential if we are to prove the soundness of a three-dimensional parallel shrinking (or thinning) algorithm.
Certain aspects of the case of (6,25) digital pictures were studied in some depth by Toulakis and Mylopoulos in [ToulakisMylopoulos73]. They called a black point \( p \) 
\[ \text{deletable} \] if the configuration of black and white points in the 3 by 3 by 3 neighbourhood of \( p \) satisfied a certain criterion, which we shall describe in greater detail later on in this chapter. They defined two sets of black points to be topologically equivalent if one set could be obtained from the other by sequentially inserting ("blackening") and deleting ("whitening") deletable points, where insertions and deletions can be arbitrarily interleaved. Their definitions of deletability and topological equivalence also apply in two dimensions, and in higher dimensions than three. In the two-dimensional case it is in fact readily confirmed that the deletable points of Toulakis and Mylopoulos are the same as the deletable points used by earlier authors for (4,8) digital pictures.

However, the possibility of inserting as well as deleting deletable black points can cause components, holes and cavities to "move" long distances away from their starting positions. As shrinking algorithms are not supposed to have this effect, the definition just given must be modified to suit our present purposes. It seems reasonable to consider a shrinking algorithm to be sound (i.e., topology preserving) only if the output black set can be obtained from the input black set by sequentially deleting deletable black points. We shall refer to this as the Modified TM definition of shrinking.

More recently, Morgenthaler [Morgenthaler81] adopted an alternative approach to shrinking. We believe the essence of his idea may be satisfactorily expressed as follows. If \( S \) and \( T \) are two sets of lattice-points such that \( S \subseteq T \) then replacement of \( T \) by \( S \) is an admissible shrinking operation in Morgenthaler's sense if: (i) each connected component of \( T \) contains exactly one component of \( S \); (ii) each component of the complement of \( S \) contains just one component of the complement of \( T \); (iii) each closed digital curve in \( T \) can be deformed within \( T \) to a digital curve in \( S \); and (iv) whenever a closed digital curve in \( S \) can be deformed within \( T \) to another closed digital curve in \( S \) than the first curve can also be deformed within \( S \) to the second curve. Of course, "closed digital curve" and "deform" can be given precise meanings. (In more technical terms conditions (iii) and (iv) state that the inclusion of \( S \) in \( T \) induces isomorphisms of the fundamental
groups of the components of 5 and 7. For the purpose of establishing that a proposed parallel shrinking algorithm is sound, it would appear that Morgenthaler's definition is easier to use than the Modified TM definition.

Morgenthaler defined three-dimensional simple points for (6,26) and (26,6) digital pictures. A black point is simple if its deletion does not change the number of components of black or white points in its 3 by 3 by 3 neighbourhood, and also does not change the "Euler characteristic" of the set of black points in that neighbourhood. (This formulation of the definition in terms of the Euler characteristic is due to [TsaoFu82]). For the moment it suffices to think of the Euler characteristic as the number of components plus the number of "cavities" minus the number of "holes" (or "tunnels") in the black set. We shall give precise definitions in Chapter 3. Morgenthaler argued that deletion of a black point is a topology-preserving operation in his sense if and only if that point is simple.

Although we showed in Chapter 4 that in the (6,26) case Morgenthaler's simple points are precisely Tournakis and Mylopoulos's deletable points, the Modified TM definition of an admissible shrinking operation is strictly stronger than the (6,26) case of Morgenthaler's definition -- while an admissible shrinking operation in the Modified TM sense certainly preserves topology in the sense of [Morgenthaler81], the converse is false.

The new interpretation of "topology preservation" we give in Chapter 4 is one that we find easier to visualize than either of the definitions we have just described. As we mentioned above, in this interpretation a shrinking operation is sound if the standard continuous analogue of the input black set can be continuously deformed in a specific and very simple way to the standard continuous analogue of the output black set. Thus our definition is very much closer in spirit than the others to the earlier two-dimensional definition given by Hilditch in [Hilditch69]. As regards practical applications, we believe our definition of shrinking will in general be no harder to use in soundness proofs than that of Morgenthaler, and will usually be more convenient than the Modified TM definition.
We apply our definition to digital picture based on many other kinds of adjacency relation in addition to $(6,26)$ and $(25,6)$. The $(6,26)$ case of our definition of an admissible shrinking operation is implied by the Modified TM definition, but it is at present unclear to us if the two definitions are in fact equivalent. The precise logical relationships between Morgenthaler's definitions of shrinking and the $(6,26)$ and $(26,6)$ cases of our definition are also unclear, but we do know that Morgenthaler's definitions do not imply ours. However, we shall show that a black point is simple in Morgenthaler's sense if and only if its deletion preserves topology in our sense. Thus in the case of sequential shrinking algorithms (in which black points are deleted one at a time, and the image is updated every time a point is deleted) the three definitions are equivalent.

A Historical Survey of Research Papers on Digital Topology

Hilditch's paper [Hilditch69] is probably best known for the thinning algorithm she described, which has attracted a good deal of attention in the literature over the years is still highly regarded today. The paper also provides a very informative introduction to two-dimensional thinning. But for our purposes the paper is of interest because of its useful discussion of plane digital topology. In particular this is one of the earliest papers to use opposite kinds of adjacency on black and white points ($B$-adjacency is used on black points and $4$-adjacency on white points), and the author gives clear reasons for doing this. (Rosenfeld [Rosenfeld70] attributes the idea to Dude et al. [Dudo+67].) [Hilditch69] is also noteworthy for explicitly defining the notion of topology preservation, as we have already mentioned.

Another interesting feature of the paper from our point of view is the author's concept of a "crossing number" $X(p)$. $X(p)$ is equal to the total number of $B$-components of black $0$-neighbours of $p$, except that if all the $4$-neighbours of $p$ are black points then $X(p) = 0$. The crossing number is so called because it can be computed by visiting the $B$-neighbours of $p$ in cyclic order, always cutting the corner between $B$-adjacent pairs of black $4$-neighbours, and counting the number of times we cross over from a black point to
a white point.

The topological significance of $X(p)$ is that deletion of a black point $p$ is a topology preserving operation (for (B,4) adjacency) if and only if $X(p) = 1$. Such points have been called deletable, inessential or simple by subsequent authors (e.g. [Kosnfeld70, Mylopoulos&Pavlidis71b, Rosenfeld75e]).

Hilditch's crossing number $X(p)$ is equal to the connectivity number $N_6^{14}(p)$ defined by Yokoi et al. in a subsequent paper [YokoiToriwakiFukumura73], where the analogous quantity $N_6^{14}(p)$ for (4,8)-adjacency is also defined. But $X(p)$ is not the same as the crossing number $X(p)$ defined by Rutovitz in [Rutovitz66]. In fact $X(p)$ is equal to twice the total number of 4-components of black B-neighbours of $p$, except that if all the B-neighbours of $p$ are black points then $X(p) = 0$. $X(p)$ is of some historical interest as "the first important measure of connectivity which produced a considerable impact on succeeding thinning algorithms ... " [Tamura78]. Indeed, $X(p)$ and $X(p)/2$ have been used in the point deletion criteria of a number of thinning algorithms since 1966 (e.g. [Deutsch71], [Deutsch72], [ArcelliSanniti78], [ArcelliSanniti81] and [ZhangSuan84]).

A number of appealing non-trivial results in digital topology were established by Rosenfeld in [Kosnfeld70]. We shall describe a selection of these. A discrete Jordan Curve Theorem was proved -- a converse was proved later, in [Rosenfeld75e]. Specifically, it was proved by a direct "digital" argument that the complement of a simple closed 4-curve is not connected. In [Rosenfeld73] this was strengthened to assert that the complement has just two 8-components.

Another result proved was that the border of a 4-connected set with an 8-connected complement is connected, and could be "tracked" by a border following algorithm. However, in both the (4,8) and (8,4) cases the border was defined to be the set of black points that are 4-adjacent to a white point and this border was proved to be B-connected. In recent literature the (4,8) border is also frequently taken to be the set...
of black points that are 8-adjacent to a white point, and 4-adjacency is used for the
border, but the result would still be true with this choice of adjacencies, as we show in
Chapter 2.

The notion of a deletable point was introduced in the (6,4) case these are just the
points \( p \) with \( X(p) = 1 \), and in the (4,8) case the deletable points are the black
points that have a white 6-neighbour and are 4-adjacent to exactly one 4-component of black
8-neighbours. Thus if \((\alpha, \beta) = (4,8) \) or \((0,1) \) then the \((\alpha, \beta) \) deletable points are just
the black points \( p \) with a white 8-neighbour whose deletion does not change the total
number of black \( \alpha \)-components in the 3 by 3 neighbourhood of \( p \). In
[MylopoulosPavlidis71b] it is shown that, given any two 4-connected sets whose
complements have the same number of 8-components, it is possible to transform one to
the other by insertion and deletion of (4,8) deletable points.

An equally proved but conceptually very important alternative characterization of a
deletable point is implied by Proposition 2 of [MylopoulosPavlidis71b], and Proposition 4.1
of [Rosenfeld73] gives a more explicit statement of the relevant result: the \((\alpha, \beta) \)
deletable points are exactly the black points \( p \) with a black \( \alpha \)-neighbour whose deletion
does not change the total number of white 8-components in the 3 by 3 neighbourhood of
\( p \). This shows that \( p \) is \((\alpha, \beta) \) deletable if and only if \( p \) becomes \((8,0) \) deletable when
the colours of all points except \( p \) are changed (so that black points become white and
vice versa).

A third characterization was used by [Gray71]: the \((\alpha, \beta) \) deletable points are the black
points \( p \) whose deletion does not change the \((\alpha, \beta) \) Euler characteristic of the set of
black points. (In two dimensions the \((\alpha, \beta) \) Euler characteristic of a set of points is
equal to the number of \( \alpha \)-components of the set plus one minus the number of
\( \beta \)-components of the complement.) Later, Yokoi et al. ([YokoiToriwakiFukumura73], eons.
9 and 10) gave a more explicit statement and proof of the correctness of this
characterization of a deletable point. In three space these three characterizations are
no longer equivalent: Mergenthaler's simple points are exactly the black points that
satisfy all three conditions.

On the problem of two-dimensional image shrinking, [Rosenfeld70] contains the result that
if a black set containing more than one point is 4-connected and the complementary white
set is 6-connected then there is a (4,8) deletable black point; so all such black sets
can be reduced (shrunk) to a single point by repeated deletion of deletable points.
Modifications to the proof were suggested that would yield the analogous result for (6,4)
adjacency. An extension was proved in [Rosenfeld73]: if the black set is 4-connected
and the white set has exactly two 8-components then the black set can be shrunk to a
simple closed 4-curve by repeated deletion of (4,8) deletable points. Alexander and
Theler proved in [Alexander-Theler71] that repeated deletion of deletable black points will
always shrink the black set to a set which does not contain any 2 by 3 arrays of black
points.

A quite deep result was proved using homology techniques in [Alexander-Theler71]. The
authors defined two boundary counts \(I(p)\) and \(J(p)\) for black points \(p\). \(J(p)\) is the number
of different boundary curves that contain \(p\), where a boundary curve is a (4-connected)
sequence of black points visited by a precisely specified "left-hand-on-wall"
border-following algorithm which tracks black points that are 8-adjacent to a white point.
Different boundary curves are detected by starting the algorithm off from different pairs
of 8-adjacent black and white points. Naturally, two boundary curves are considered to be
the same if one is a cyclic permutation of the other.

The notion of a boundary curve as a sequence of black points should be contrasted with
the concept (e.g. [Rosenfeld70]) of a border as a set of black points. Thus if the
black points are arranged in a simple closed 4-curve \(C\) then there are two distinct
boundary curves, one going clockwise around \(C\), the other going anticlockwise around \(C\). On
the other hand if the black points are arranged in an \(n\) by \(n\) square then there is just
one boundary curve -- this goes anti-clockwise around the border because the tracking
algorithm is “left-hand-on-wall”. This concept of a boundary curve is closely related to 
the bicurves of [Rosenfeld74], which consist of ordered pairs of adjacent points, the 
first of which is a black point and the second of which is a white point. However 
4-adjacency between black and white points is used in the definition of a bicurve, whereas 
Alexander and Thaler use 8-adjacency between black and white points.

A boundary curve can pass through the same point more than once (consider the central 
uode of a figure of eight). The authors define I(p) to be the result of adding up the 
number of times p occurs on each of the boundary curves that pass through p. Thus I(p) 
\geq J(p). The authors showed that I(p) = 1 + (the increase in the total number of black 
4-components of the 3 by 3 neighbourhood of p when p is deleted), except that if p has 
no black 4-neighbours then I(p) = 1. (*). (Yokoi et al. gave a very short proof of this 
in Theorem 8 of [YokoiToriwakiFukumura73].)

Let c(X) denote the number of 4-components of X. The theorem we referred to above is 
that if P is the set of black points and a point p in P is B-adjacent to a white point 
then \( c(P\setminus\{p\}) = c(P) + I(p) - J(p) \).

The reader may be interested in our own approach to the above theorem. For all black 
points p define \( J'(p) \) to be the number of white 8-components that are 8-adjacent to p. 
Then, assuming p has at least one black 4-neighbour and at least one white 8-neighbour 
(otherwise the theorem is trivial), \( c(P\setminus\{p\}) - c(P) = J'(p) + 1 \). For the left 
hand side is equal to the increase in the Euler characteristic (no. of components - no. 
of handles) of the black points when p is deleted, and by (*) the right-hand side is equal 
to the increase in the Euler characteristic of the set of black points in the 3 by 3 
neighbourhood of p when p is deleted. But these two increases must be equal because 
the change in Euler characteristic is “locally determined” (in virtue of the well-known 
formula \( X = \#edges + \#vertices - \#faces \)).

It follows from this that Alexander and Thaler’s theorem is equivalent to the identity
\( J(p) = J'(p) \), which says that their border following algorithm "works correctly" -- i.e. each white \( \beta \)-component that is \( \beta \)-adjacent to \( p \) determines exactly one boundary curve through \( p \). This result is deep, but nevertheless "intuitively obvious". One way to prove it is by extending the sequence (7) in [AlexanderThaler71] to the left by the term \( H_s(P \cdot 0(0)) \), and then applying the result \( \text{rank}(\text{image } \beta) = J(0)-1 \) (proved in the same section of the paper), together with the standard theorem from plane topology that \( \text{rank}(H_s(P)) = \text{no. of holes in } P \). Alternatively one could prove the result from first principles by arguing along the lines of Section 6 in [Rosenfeld70].

The results \( c(P \setminus \{p\}) = c(P) + I(p) - J(p) \), and \( J(p) = J'(p) \) yield a non-trivial characterization of deletable points: \( p \) is \((0,\beta)\) deletable if and only if deletion of \( p \) changes neither the total number of black \( \beta \)-components nor the total number of white \( \beta \)-components. This result can also be deduced without much difficulty from Proposition 7 in [Rosenfeld70].

[TourlakisMynopoulos73] represents a significant advance in digital topology. It is the first paper to give an in-depth study of three- (and higher) dimensional digital topology. The authors considered the digital pictures based upon the \((4,6)\) and \((6,26)\) adjacency relations, and the higher dimensional analogues of these. They gave a definition of a deletable point that applies to three- or higher dimensions as well as applying to digital pictures in the plane.

Let \( d \) be the dimension of the ambient space. For any \( P \in \mathbb{Z}^d \) and all \( 0 \leq r \leq d \) let \( K^r(P) \) be the set of all \( r \)-dimensional unit hypercubes whose vertices all belong to \( S \). (\( S \) \( K^0(S) = S \)). Let \( K^r(P) = \bigcup K^r(P) \) \( 0 \leq r \leq d \). For each \( x \) in \( P \) the authors defined \( B(x,P) \) to be the union of all \( c \) in \( K(P) \) such that \( x \notin c \) but there exists \( k \) in \( K(P) \) such that \( x \in k \) and \( c \subseteq k \). Plainly \( B(x,P) \) is a subset of the boundary of the \( d \)-dimensional hypercube with sides of length 2 and centre at \( x \).

In the cases \( d = 2 \) and \( d = 3 \) a point \( x \) is deletable in \( P \) in the sense of Tourlakis and
Mylopoulos if and only if \( B(x, P) \) is simply connected and \( B(x, P) \) is not the entire surface of the \( Z \) by \( Z \) square \((d = 2)\) or \( Z \) by \( Z \) by \( Z \) cube \((d = 3)\) with centre \( x \). This is not their definition, but it is equivalent to to their definition by the "(a) is equivalent to (c)" part of Proposition 3.8 in their paper. (Incidentally, we suspect that a part of condition (b) has been omitted in the statement of this proposition -- it looks as if condition (b) should read "\( H_2(P) \cong H_2(P\setminus\{x\}) \) for all \( k > 0 \), and such isomorphisms are induced by the inclusion of \( P\setminus\{x\} \) in \( P \)." The statement of Corollary 3.10 requires analogous modifications.)

Many substantial results are proved in the paper. From the viewpoint of this thesis the most interesting result is Lemma 3.2, which asserts that if \( x \) is deletable in \( P \) then \( UK(P) \) collapses to \( UK(P\setminus\{x\}) \). (Topologists might be more interested in their Theorem 4.1, which shows that if \( x \) is deletable in \( P \) then the regular neighbourhood of \( UK(P) \) in \( S^d \) is ambient isotopic in \( S^d \) to the regular neighbourhood of \( UK(P\setminus\{x\}) \) in \( S^d \).) The definition of topological equivalence for digital pictures proposed in the paper has of course already been mentioned.

In Chapter 4 we generalize Lemma 3.2 to the (25.6), (18.6) and (6.19) digital pictures, and we show that Tourlakis and Mylopoulos's deletable points are the same as the \((4,0)\) deletable points of [Rosenfeld70] in two dimensions, and to the (6.26) simple points of [Morgenthaler81] in three dimensions. (The first of these results is quite easy).

Comparison of our work with [TourlakisMylopoulos73] will highlight the fact that we do not discuss homology and homotopy. The reason is that -- unlike Tourlakis and Mylopoulos -- we confine our attention to two- and three-space, and as a consequence the simpler notion of the Euler characteristic is already sufficiently powerful to prove our results. In fact topologists will be aware that all the homology groups of a polyhedron \( Q \) in \( S^3 \) can be computed from \( X(0) \) (assuming we know the number of components in \( Q \) and \( S^3\setminus\{0\} \). It is shown in [KongRoussos65b] that one can also determine whether or not a polyhedron \( Q \) in \( S^3 \) is simply-connected from the value of \( X(0) \).

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[Rosenfeld75a] proves a useful result in the theory of two-dimensional parallel thinning, which is most conveniently stated in terms of some new but very natural terminology. A black point with coordinates \((x,y)\) is said to be a north border point if its \(4\)-neighbour \((x,y+1)\) is a white point. The set of black points is said to be a simple \(8\)-arc if each black point has exactly two black \(8\)-neighbours, with the exception of two points (called end points) each of which has exactly one black \(8\)-neighbour. The result can now be stated as follows:

Let \(\Phi\) be an algorithm which deletes in parallel all north border black points that satisfy given “locally checkable” criteria (i.e. criteria which depend only on the black-white point configuration in the \(3\times3\) neighbourhood). Then \(\Phi\) is \((8,4)\) topology preserving for all possible inputs and leaves any simple \(8\)-arc unchanged if and only if the point deletion criteria are such that every point deleted is an \((8,4)\) deletable point with at least two black \(8\)-neighbours.

The significance of this theorem is that it completely solves the problem of \((8,4)\) topology preservation for a large class of parallel thinning algorithms, namely those consisting of passes which delete selected border points from one side (north, south, east or west), leaving simple \(8\)-arcs unchanged, in which the point deletion criteria are locally checkable and never change within a pass. (In this context a pass is the time between successive “updates” of the image.) When applying the theorem to practical algorithms account has to be taken of the fact that if noise removal and thinning are ever done in the same pass, then that pass will not necessarily preserve topology.

We observe that the result is invalid if \(\Phi\) deletes border points from more than one side in parallel. Consider the following example, in which the 1’s are the black points, and \(p = q = 1\).

\[
\begin{array}{ccc}
1 & 1 & p & 1 & 1 \\
1 & 0 & q & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
Here \( p \) is a north border point and \( q \) is a west border point. Plainly both \( p \) and \( q \) are \((8,4)\) deletable. But their simultaneous deletion will merge two white \( 4 \)-components.

Later, Arcelli gave a quick test for determining whether or not a north-border point is \((8,4)\) deletable [Arcelli79]. He showed that a north-border black point \( p \) is \((8,4)\) deletable if and only if \((W \cdot S \cdot E + -W \cdot N \cdot N + -N \cdot E \cdot E + -E \cdot S \cdot S + -S \cdot W \cdot W) = 0\). Here \( W = 0 \) or 1 according as the west neighbour of \( p \) is white or black, and similarly for the other compass directions (SW refers to the south-west neighbour, etc.). Of course, "+", "." and "-" denote boolean disjunction, conjunction, and negation respectively.

One result established in [RaoDanielssonKrusa78] is reminiscent of the main theorem of [Rosenfeld75a]. To state this result, define a template to be any 3 by 3 array of black, white or grey points, in which the central point is black. It is plain that a collection of templates corresponds to a locally checkable point deletion criterion in a natural way (i.e., a black point is to be deleted if and only if its 3 by 3 neighbourhood matches one of the templates, where a grey point in the template is a "don't care" which matches either a white point or a black point). Call a point deletion criterion a single-template criterion if it is the criterion corresponding to some single template.

The result we referred to is that any algorithm which deletes in parallel all black points that satisfy a given single-template criterion is \((8,4)\) topology preserving for all possible inputs if and only if the criterion is such that every point deleted is \((8,4)\) deletable. (This follows from Corollary 1 and Theorem 2 in [RaoDanielssonKrusa78].) Rao et al. gave a boolean expression which can be used in an analogous fashion to Arcelli's expression (described above) for ascertaining whether or not a single-template criterion is indeed satisfied only by \((8,4)\)-deletable points.

Much work was done by Morgenthaler [Morgenthaler81] on the problem of generalizing the main theorem of [Rosenfeld75a] to three dimensions. [Morgenthaler81] also introduces three-dimensional simple points, and a new definition of topology preservation for shrinking algorithms. These have already been described.
[TsaoFu82] reports three interesting facts about simple points of digital pictures. The first is that in the definition of a (26,6) simple point p the condition "the number of white 6-components in the 3 by 3 neighbourhood does not change when p is deleted" is implied by the other defining conditions. One can deduce (by symmetry) that, in the definition of (6,26) simple points, the condition "the number of black 6-components in the 3 by 3 neighbourhood does not change when p is deleted" is redundant. We give a new proof of this result in Chapter 4 (Theorem 8).

In his original definition of a simple point, Morgenthaler stipulated that for a black point p to be simple, it must satisfy four conditions, two of which were that deletion of p must not change the number of "holes" in the black set and also must not change the number of "holes" in the white set. Tsao and Fu saw that these two conditions are in fact equivalent. An independent proof of the same result is given in Chapter 3 (Proposition 1).

To state their third discovery, suppose that p is a black point whose deletion does not change the Euler characteristic of the set of black points. Let $P_x$, $P_y$, and $P_z$ denote the x, y and z coordinate planes that contain p. The result reported is that p is (26,6) simple if p is (8,4) deletable with respect to the black points on any two of $P_x$, $P_y$, and $P_z$. By symmetry, p is (6,26) simple if p is (4,8) deletable on at least two of $P_x$, $P_y$, and $P_z$. (The converse is false, as the authors showed.) One can prove that an analogous result holds for (18,6) and (6,18) digital pictures.

In [TsaoFu82] the "hole preservation" condition in the definition of a simple point is expressed in terms of the Euler characteristic; we also use this formulation.

This ends our literature survey, which is admittedly incomplete. As time was short, only those works that seemed to us to be especially relevant to our own research have been included. Moreover, a number of works which we found equally useful have been omitted, in most cases because they are of particular importance to certain parts of the thesis and are described there.
Chapter 2 - A THEORY OF BINARY DIGITAL PICTURES

Introduction

Rosenfeld's paper [Rosenfeld81] provides a very clear exposition of the fundamental concepts of three-dimensional digital topology. However, only two kinds of adjacency relation are considered in [Rosenfeld81]: either 6-adjacency is used for the 'objects' and 26-adjacency for the 'background', or 26-adjacency is used for the 'objects' and 6-adjacency for the 'background'. Yet it is clear that the concepts like digital paths and digital components make sense for a wide variety of adjacency relations. The first goal of this chapter is to present a simple unified approach to digital pictures which places no artificial restrictions on the adjacency relations used.

Morgenthaler and Rosenfeld suggested in [MorgenthalerRosenfeld81] that we might sometimes wish to define adjacency between points 'which are not even "near" each other.' It might be possible to extend our new theory so as to allow this. In another direction Mylopoulos and Pavlidis showed in [MylopoulosPavlidis71a,71b] that many of the basic concepts of digital geometry remain valid when the conventional rectangular grid is replaced by the Cayley diagram (c.f. [Bollobas79], ch 8) of any finite presentation of an abelian group - they called such group presentations discrete spaces. It is likely that the ideas introduced in this chapter can be applied to many of these discrete spaces.

Our work on general binary digital pictures has interesting corollaries for particular adjacency relations. Not only can we obtain Propositions 5 and 6 of [Rosenfeld81] but we also find that similar results hold for every pair of "pure" adjacency relations (6-, 18- or 26-adjacency) other than (6,6). These and related results proved in this chapter can be regarded as establishing the fundamental separation and connectivity properties of digital borders. We regard a proof of these results as our second goal. That these results are fairly deep becomes apparent when we note that the two dimensional analogues of both results fail on the surface of a cylinder (when $Z^2$ is replaced by $Z_n \times Z$ where $Z_n$ denotes the integers modulo n). The point is that if n is large then $Z_n \times Z$ is locally
indistinguishable from $Z^r$, which shows that the two propositions express global properties of Euclidean space. (Geometric topologists will recognise that the propositions express discrete versions of two ‘Poincaré-Brouwer properties’ [Wilder49]).

To establish the validity of such results we must use a ‘global’ proof method: purely local methods (such as straightforward induction on the number of points in $S$, or simple graph-theoretic arguments) are unlikely to suffice. Our method of attack involves applying techniques from continuous topology to the continuous analogues of binary digital pictures. We do this in the ‘IV implies I’ part of the proof of Theorem 2.

Finally we point out some of the drawbacks of using any single elementary adjacency relation ($4^-$, $6^-$, $25^-$ etc.) on the conventional square and cubic grids. In order to overcome these difficulties most other workers have resorted to the use of different adjacency relations for black and white points. Our investigations suggest that from a theoretical standpoint, and probably also for some practical purposes, it may be a better choice to use a two-dimensional hexagonal lattice or a three-dimensional face-centred cubic lattice equipped with the corresponding ‘nearest neighbour’ adjacency relations.

This chapter is structured in such a way that it is possible to omit the sections relating to continuous analogues, and the proofs of Theorems 2 and $2^r$, provided that the ‘I is equivalent to II’ part of these theorems is assumed without proof. The statement and proof of Proposition 3 may also be omitted on a first reading; however, the Corollary to Proposition 3 is used in the proof of Proposition 4.

**Basic Concepts**

**Simply-Connected Sets**

It will emerge that Propositions 5 and 6 in [Rosenfeld81] are valid because $R^2$ and $R^3$ are both simply-connected.
A connected subset $Y$ of $\mathbb{R}^2$ or $\mathbb{R}^3$ is said to be simply-connected if it has no 'holes'. (A solid cube is simply-connected but a solid torus is not.) Equivalently, $Y$ is simply-connected if given any two curves in $Y$ with the same endpoints we can transform one curve into the other by means of a 'continuous deformation' during which both endpoints remain fixed and the rest of the curve remains in $Y$. The precise definition is as follows:

A curve in a subset $Y$ of $\mathbb{R}^n$ is a continuous map $\gamma : [0,1] \to Y$. The curve $\gamma$ is said to be a curve in $Y$ from the point $\gamma(0)$ to the point $\gamma(1)$. The trace of a curve $\gamma$ is another name for the image of $\gamma$. A connected subset $Y$ of $\mathbb{R}^n$ is said to be simply-connected if given any two points $p$ and $q$ in $Y$ and any two curves $\gamma_0$ and $\gamma_1$ each of which is a curve in $Y$ from $p$ to $q$, we can find a continuous map $h : [0,1] \times [0,1] \to Y$ such that for all $s$ and $t$ in $[0,1]$ we have

(i) $h(s,0) = \gamma_0(s)$  
(ii) $h(s,1) = \gamma_1(s)$  
(iii) $h(0,t) = p$  
(iv) $h(1,t) = q$

The 'continuous deformation' $h$ is called a fixed endpoint homotopy of $\gamma_0$ onto $\gamma_1$. It is customary to think of $t$ as 'time'.

We show in the Appendix to this chapter that the definition we have just given is equivalent to the standard definition.

We asserted above that a solid torus is not simply connected. This is intuitively clear but not so easy to prove. However, it is a corollary of Theorem 2 below.

**Boundaries**

We define the frontier of a set $X \subseteq \mathbb{R}^n$ to be the closure of $X$ in $\mathbb{R}^n$ minus the interior of $X$ in $\mathbb{R}^n$. We write $\text{Fr } X$ for the frontier of $X$ in $\mathbb{R}^n$, $\text{cl } X$ for the closure of $X$ in $\mathbb{R}^n$ and $\text{int } X$ for the interior of $X$ in $\mathbb{R}^n$.

If $M$ is an $n$-manifold with boundary then $\text{Bd } M$ denotes the manifold boundary of $M$. We shall generalize this definition slightly in Chapter 5.
EXAMPLE: If $D$ is a disc in $R^3$ then $Bd \ D$ is the perimeter circle of $D$ and $Fr \ D = D$. 

Elementary Terminology

$Z$ denotes the set of integers and $R$ denotes the set of real numbers; $Z$ is regarded as a subset of $R$. Thus $R^n$ denotes Euclidean $n$-space and $Z^n$ is the set of all lattice points in Euclidean $n$-space. We use the term 'lattice point' to denote a point in $Z^2$ or $Z^3$. We identify $Z^n$ with the set of points in $R^n$ that have integer coordinates (in the obvious way).

If $W \subseteq Z^n$ then $W^c$ denotes the complementary set $Z^n \setminus W$ (here $n = 3$ in the sections relating to three-dimensional digital pictures and $n = 2$ in the sections on two-dimensional pictures). A unit cell is a closed unit cube (in 3D) or a closed unit square (2D) whose corners are all lattice points. (Note that a unit cell is a connected subset of $R^3$ or $R^2$ and not a set of lattice points.) A window is any union of unit cells.

If $K$ is a unit cell and $5xZ^n$ then $K^S$ can be mapped by rotation or reflection onto one of the twenty-two sets shown in Figure 1 (a proof is given in the Appendix to this chapter, but in any event it is readily confirmed that Figure 1 does exhaust all possibilities.). We shall say that the pair $(K,S)$ is of type $n$ if $K^S$ can be mapped by rotation or reflection onto the $n$th set in Figure 1.

Binary Digital Pictures

The following definition is the basis of our new approach to digital topology:

A three-(two-)dimensional binary digital picture is a pair $(A,S)$ where $S$ is any subset of $Z^3$ ($Z^2$) and $A$ is any symmetric binary relation on $Z^3$ ($Z^2$) that satisfies the axioms (i) and (ii) below. We shall say that $x$ is $A$-adjacent to $y$ if (and only if) $(x,y) \in A$. The axioms $A$ must satisfy are:

(i) If $x$ and $y$ are $6$-(4-)adjacent then $x$ and $y$ are $A$-adjacent.

(ii) If $x$ and $y$ are $A$-adjacent then $x$ and $y$ are $26$-(8-)adjacent. 

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FIGURE 1: The Twenty-two Different Types of Unit Cell
The set $S$ is usually derived from pictorial data, but the adjacency relation $R$ is chosen by the user. In the above definition the adjacency relation is explicitly included as part of a binary digital picture; we have found it helpful to think in this way. However, we are not the first to incorporate adjacency relations into the mathematical structure of a digital picture. Previous authors did so implicitly when using terms like 'connectedness in the sense of the background'.

Nevertheless, the definition just given represents a slight but significant departure from the usual conceptual framework of digital geometry, because the adjacency relation has been freed of all dependence on the set of object points: there is no longer any notion of 'adjacency in the $S$ (or $S^2$) sense'.

In the rest of this thesis (A.5) will be a two- or three-dimensional binary digital picture.

More Elementary Terminology

In this section and the next we adapt the terminology of [Rosenfeld81] to our new definition of digital pictures.

If two points are $A$-adjacent then each is called an $A$-neighbour of the other. A point is $A$-adjacent to a set if it is $A$-adjacent to some member of that set. Two disjoint sets of lattice points will be said to be $A$-adjacent if there is a point in one subset that is $A$-adjacent to some point in the other. An $A$-path is a sequence of distinct lattice points such that any two consecutive points in the sequence are $A$-adjacent. If each point in an $A$-path belongs to some set $T$ then we shall call the path an $A$-path in $T$. An $A$-path whose first point is $x$ and whose last point is $y$ will be called an $A$-path from $x$ to $y$, or alternatively an $A$-path that links $x$ to $y$. We shall say $S$ is $A$-connected if every pair of points in $S$ is linked by an $A$-path. An $A$-connected subset of $S$ that is not $A$-adjacent to any other point in $S$ will be called an $A$-component of $S$. Thus $S$ is $A$-connected iff $S$ contains just one $A$-component. If $S$ and $T$ are disjoint sets of lattice points then the $(A.S)$-border of $T$ (or, alternatively, the $A$-border of $T$ with
respect to $S$) is defined to be the set of all points in $I$ that are $A$-adjacent to a point in $S$.

If $(A,S)$ is a binary digital picture then we refer to $S$ as the set of object points of the picture, we refer to $S^c$ as the set of background points of the picture, and we refer to $A$ as the adjacency relation of the picture. Note that we use the term "background point" in a weaker sense than some authors, who prefer not to use the term for points in a bounded $A$-component of $S^c$. One justification for this is that we have chosen to work in the "wrap-around" spaces $Z^2_n$ and $Z^3_n$ in Chapters 3 and 4, in which all components are finite.

If $A$ is an adjacency relation on $Z^2$ or $Z^3$ and $X$ is a window then $A(X)$ denotes the adjacency relation on the set of lattice points in $X$ such that $x$ is $A(X)$-adjacent to $y$ iff there is a unit cell in $X$ which contains both $x$ and $y$, and $x$ is $A$-adjacent to $y$.

When we are considering digital pictures in 3-space, we use the Greek letters $\alpha$, $\beta$, $\gamma$ and $\delta$ to denote one of the numbers 6, 18, or 26. When we are considering digital pictures in the plane we use $\alpha$, $\beta$, $\gamma$ and $\delta$ to denote 4 or 8. We shall frequently want to use the notions defined in the previous paragraph in the special cases where $A$ is the $\alpha$-adjacency relation for some $\alpha$; it will then be very convenient to use the prefix 'a' in place of 'A'. Thus we might refer to an 1B-component ($=\text{an } A\text{-component where } A \text{ is the } 1B\text{-adjacency relation}$), or 'the $(B,S)$-border of $T$' ($=\text{the } (A,S)\text{-border of } T \text{ where } A \text{ is the } B\text{-adjacency relation}$). This use of the numbers 6, 18, 26, 4 and 8 as prefixes is fully consistent with the usage established by previous authors.

If $K$ is any unit cell in $R^2$ then we define $K^*$ to be the union of $K$ with the four other unit cells that have an edge in common with $K$. Similarly, if $K$ is any unit cell in $R^3$ then we define $K^*$ to be the union of $K$ with the six other unit cells that have a face in common with $K$. 

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We define \((a, b, o, s)\) to be the digital picture \((A, S)\) in which \(A\) is the adjacency relation on \(Z^2\) or \(Z^3\) in which two points \(x, y\) are \(A\)-adjacent iff either \(x\) and \(y\) are \(u\)-neighbours in \(S\), or \(x\) and \(y\) are \(\beta\)-neighbours in \(S^c\), or \(x\) and \(y\) are \(S\)-neighbours and exactly one of \(x\) and \(y\) belongs to \(S\). We also use \((a, b, x, s)\) to denote this adjacency relation. (It should always be apparent from the context whether \((a, b, o, s)\) denotes a digital picture or an adjacency relation.) For brevity, we use \((a, b, s)\) to denote \((a, b, o, s)\) and we use \((o, s)\) to denote \((a, o, o, s)\).

**Adjacency Graphs**

In this section we introduce a (rather trivial) generalization of a notion introduced in [Buneman69] and more thoroughly investigated in [Rosenfeld74].

Let \(X\) be any window. The \(X\)-adjacency graph of \((A, S)\), denoted by \(\text{adj}(A, X, S)\), is a (possibly infinite) bipartite graph each of whose vertices is a whole \(A\langle X\rangle\)-component of \(S^nX\) or \(S^nX\). The first vertex class of \(\text{adj}(A, X, S)\) consists of the vertices corresponding to \(A\langle X\rangle\)-components of \(S^nX\) (these are called the \(S\)-vertices of \(\text{adj}(A, X, S)\)); the second vertex class consists of the vertices corresponding to \(A\langle X\rangle\)-components of \(S^nX\) (these are called the \(S^c\)-vertices of \(\text{adj}(A, X, S)\)). An \(S\)-vertex \(x\) and an \(S^c\)-vertex \(y\) are joined by an edge of \(\text{adj}(A, X, S)\) if and only if the components represented by \(x\) and \(y\) are \(A\langle X\rangle\)-adjacent.

The adjacency graph of \((A, S)\), which we denote by \(\text{adj}(A, S)\) is defined to be the (possibly infinite) graph \(\text{adj}(A, R^3, S)\) (3D case) or \(\text{adj}(A, R^2, S)\) (2D case). Thus the \(S\)- and \(S^c\)-vertices of \(\text{adj}(a, b, o, s)\) are respectively the \(\alpha\)-components of \(S\) and the \(\beta\)-components of \(S^c\); and an \(S\)-vertex is joined to an \(S^c\)-vertex iff the corresponding components are \(X\)-adjacent.

Following [Rosenfeld81] we observe that if \(\alpha\) and \(\beta\) are not both equal to \(6\) (or, in the 2D case, not both equal to \(4\)) then an \(\alpha\)-component of \(S\) and a \(\beta\)-component of \(S^c\) are
6-(4-)adjacent iff they are 2E-(B-)adjacent. So unless \( n = k = h = 4 \) we can suppress the \( 8 \) in 'adj(0,8,8,5)' and just write 'adj(0,8,5)'. (Note that adj(8,26,5) and adj(26,8,8) are just the adjacency graphs considered in [Rosenfeld81].)

If \( x \) is a vertex of \( \text{adj}(A,X,S) \) then we use the notation \( \text{COMP}(x) \) to denote the \( A(X) \)-component corresponding to the vertex \( x \). (Strictly speaking there is no difference between \( x \) and \( \text{COMP}(x) \), but we use the latter notation whenever we are interested in the 'internal structure' of \( \text{COMP}(x) \).)

Normal Digital Pictures

DEFINITION

We shall say that a binary digital picture \((A,S)\) is normal if \( \text{adj}(A,X,S) \) is a tree for every unit cell \( K \). If \((A,S)\) is normal then we shall say that \( S \) is \( A \)-normal.

There are some adjacency relations \( A \) such that \((A,S)\) is a normal binary digital picture for any set of lattice points \( S \). The simplest examples are the 1B- and 2E-adjacency relations (in 3D) and the 8-adjacency relation (in 2D). On the other hand there are some sets \( S \) which have the property that \((A,S)\) is normal for all adjacency relations \( A \) that satisfy the conditions (i) and (ii) in the definition of a binary digital picture. In fact any \( S \) such that for every unit cell \( K \) one of the sets \( S^{NK} \) and \( S^{N\bar{K}} \) is \( S \)-connected will have this property.

Note also that if one of \( \alpha \) and \( \beta \) is not equal to 6 (4 in the 2D case) then for all \( K \) and \( S \) either \( S^{NK} \) is \( \alpha \)-connected or \( S^{N\bar{K}} \) is \( \beta \)-connected; so if \( \alpha \) and \( \beta \) are not both equal to 6 (4) then for all choices of \( \& \) the binary digital picture \((\alpha,\beta,\&,S)\) is normal.
Continuous Analogues

In this section we shall assume that we are working in three dimensions. Definitions of two-dimensional digital representations and continuous analogues are obtained by substituting \( \mathbb{R}^2 \) and \( \mathbb{Z}^2 \) for \( \mathbb{R}^3 \) and \( \mathbb{Z}^3 \).

Let \( X \) be any window. A polyhedral set in 3-space is a set which is a locally finite union of discrete points, closed straight line segments, closed triangles and closed tetrahedra (here 'locally finite' means that every bounded region meets only finitely many of the sets in the union). A continuous analogue of a binary digital picture \((A, S)\) relative to \( X \) is a polyhedral set \( C \subseteq X \) which satisfies the following conditions:

(i) \( S \cap X = C \cap \mathbb{Z}^3 \)

(ii) Let \( K \) be any unit cell contained in \( X \). If \( F \) is any connected component of \( K \cap C \) then \( F \cap \mathbb{Z}^3 \) is a union of \( A \)-components of \( S \cap K \) and is contained in one \( A(K') \)-component of \( S \cap K \cap X \)

(iii) Let \( K \) be any unit cell contained in \( X \). If \( B \) is any connected component of \( K \setminus C \) then \( B \cap \mathbb{Z}^3 \) is an \( A \)-component of \( S \cap K \)

(iv) If \( D \) is a face or edge of a unit cell \( K \) and \( K \) is contained in \( X \) then each connected component of \( D \cap C \) and of \( D \setminus C \) contains a corner of \( K \)

(v) If \( K \) is any unit cell contained in \( X \) and \( F \) and \( B \) are connected components of \( K \cap C \) and \( K \setminus C \) respectively then \( F \cap B \) meets \( F \cap B \) iff \( F \cap \mathbb{Z}^3 \) is \( A \)-adjacent to \( B \cap \mathbb{Z}^3 \)

Observe that in (v) the condition "\( F \cap B \) meets \( F \cap B \)" is equivalent to "\( F \cup B \) is connected". A very reasonable alternative condition to (v) in this definition is the following: (v)' If \( x \) is a corner of the unit cell \( K \) belonging to \( S \) and \( B \) is a connected component of \( K \setminus C \) then \( x \in F \cap B \) iff \( x \) is \( A \)-adjacent to \( B \cap \mathbb{Z}^3 \). In fact, Theorem 2 remains true if we replace (v) by (v)'.

The following proposition expresses an important property of continuous analogues; essential use will be made of this property in the proof of our principal result (Theorem 2):
PROPOSITION 0

Let $C$ be a continuous analogue of the binary digital picture $(A, S)$ relative to the window $X$. Then:

(i) If $F$ is any connected component of $C$ then $F \cap 2^2$ is an $A \langle X \rangle$-component of $S \cap X$.

(ii) If $B$ is any connected component of $X \setminus C$ then $B \cap 2^2$ is an $A \langle X \rangle$-component of $S \setminus X$.

PROOF

We shall prove (i): a proof of (ii) is obtained by substituting the terms in square brackets $[ \ldots ]$ for the terms that immediately precede the brackets. Let $\langle x_n, x_{n-1}, \ldots, x_2, x_1 \rangle$ be an arbitrary $A \langle X \rangle$-path in $X$. Then for every $0 \leq i \leq m$ there exists a unit cell $K_i$ in $X$ that contains both $x_i$ and $x_{i+1}$. So if the $x_i$ all belong to $S \setminus S^2$ then by (ii)

(iii) in the definition of a continuous analogue they all belong to the same connected component of $C \setminus (X \setminus C)$. This shows that if two points $u$ and $v$ belong to the same $A \langle X \rangle$-component of $S \cap X$ then they belong to the same connected component of $C \setminus (X \setminus C)$. In order to establish part (i) [ part (ii) ] of Proposition 0 it remains to show that if $x$ and $y$ are any two points in $2^2$ which belong to the same connected component of $C \setminus (X \setminus C)$ then $x$ and $y$ both belong to the same $A \langle X \rangle$-component of $S \cap X$ [ $S \setminus X$ ].

So let $x$ and $y$ be two such points. Let $P$ be the point-set of a polygonal arc in $C \setminus (X \setminus C)$ from $x$ to $y$, and let $p(t)$ denote the (unique) point on $P$ whose distance from $x$, when measured along $P$ is exactly $t$. Let $l$ be the total length of $P$. Thus $p(0) = x$ and $p(l) = y$. Define a finite sequence $\langle t_0, t_1, \ldots, t_n \rangle$ of real numbers in accordance with the following rules (the value of $n$ is determined by rule 3):

1) $t_0 = 0$

2) If $t_i < l$ then $t_i$ is the greatest real number such that $\langle p(t) | t \leq t_i \rangle$ is contained in a single unit cell

3) $t_n = l$

For every $i \leq n$ pick a unit cell that contains $\langle p(t) | t \leq t_i \rangle$. Call this cell $K_i$. Then, for all $1 \leq i \leq n$, $p(t_i)$ and $p(t_{i+1})$ both belong to the same connected component of $C \setminus K_i$.1
\( |K_i \setminus C| \). For each \( i \) define \( u_i \) as follows:

1. If \( \mu(t_i) \in \mathbb{Z}^3 \) then \( u_i = \mu(t_i) \)

2. If \( F \) is a face or edge of \( K_i \), such that \( \mu(t_i) \) belongs to the relative interior of \( F \) then by (iv) in the definition of a continuous analogue there is a point in \( \mathbb{R}^3 \) that belongs to the same connected component of \( F \cap C \) as \( \mu(t_i) \); define \( u_i \) to be such a point.

Hence if \( 1 \leq i \leq n \) then \( u_i \), and \( \mu(t_i) \) both belong to the same connected component of \( \mathbb{C}NK_{i+1}(K_i \setminus C) \). If \( 0 \leq i \leq n-1 \) then it follows from our construction of the \( u_j \) that \( u_{i+1} \), \( \mu(t_{i+1}) \), \( \mu(t_i) \) and \( u_i \) all belong to the same connected component of \( \mathbb{C}NK_{i+1}(K_i \setminus C) \). So by (ii) [ (iii) ] in the definition of a continuous analogue both \( u_i \) and \( u_i \) belong to the same \( \text{A}(X) \)-component of \( \mathbb{S}NK_{i+1}(\mathbb{S}^nK_i \setminus C) \). Therefore \( x (= u_n) \) and \( y (= u_0) \) belong to the same \( \text{A}(X) \)-component of \( \mathbb{S}NX(\mathbb{S}^n\mathbb{X}) \), as required. \( \blacksquare \)

In the three-dimensional case Theorem 2 will provide a useful necessary and sufficient condition for continuous analogues to exist.

**The Three-Dimensional Case**

In this section \((A,S)\) will always denote a three-dimensional binary digital picture.

The following proposition gives an intuitive understanding of the geometric significance of normality.

**PROPOSITION 1**

(i) If \((A,S)\) is normal then so is \((A, S^2)\).

(ii) Suppose \( \text{adj}(A,X,S) \) is a tree and suppose \( W \) is a subset of \( S \) such that \( W \cap X \) is a union of \( \text{A}(X) \)-components of \( S \cap X \). Then \( \text{adj}(A,X,W) \) is a tree. (Hence if \( T \) is any \( A \)-normal set then any union of \( A \)-components of \( T \) is \( A \)-normal.)
(iii) Suppose $\text{adj}(A,K,S)$ is a tree, where $K$ is a unit cell. Then either $S^0K$ is $A$-connected or $S^0K$ is $A$-connected, or the A-components of $S^0K$ and $S^0K$ are as shown in Figure 2.

(iv) Suppose $(A,S)$ is normal. Let $X$ be any window. Then an $A(X)$-component of $S^0X$ is $A(X)$-adjacent to an $A(X)$-component of $S^0X$ only if those two $A(X)$-components are $S$-adjacent.

(v) If the $(A,S^0)$-border of $S$ is $A$-normal then so is $S$ itself.

**Proof**

(i) is trivial.

(ii) Let $C$ be an $A(X)$-component of $S^0X$. Let $v$ be the $S$-vertex of $\text{adj}(A,X,S)$ that corresponds to $C$. Then $\text{adj}(A,X,S\setminus C)$ is obtained from $\text{adj}(A,X,S)$ simply by identifying $v$ and all its neighbours. Hence if $\text{adj}(A,X,S)$ is a tree then so is $\text{adj}(A,X,S\setminus C)$. But $W$ can be obtained from $S$ by removing a suitable collection of $A(X)$-components of $S$. Hence $\text{adj}(A,X,W)$ is a tree.

(iii) Suppose $K$ contains two $A$-components $C$, $C'$ of $S^0K$ and two $A$-components $B$, $B'$ of $S^0K$. Then a fortiori neither $S^0K$ nor $S^0K$ is $S$-connected, so $(K,S)$ must be of type 8, 12, 13, 14 or 15. Now if $(K,S)$ is of type 8, 12 or 13 then each point in $S^0K$ is $S$-adjacent to every $S$-component of $S^0K$, so each of $C$, $C'$ is $A$-adjacent both to $B$ and to $B'$: this implies that $\text{adj}(A,K,S)$ is not a tree, which contradicts the normality of $(A,S)$. Similarly, $(K,S)$ cannot be of type 15 (by symmetry with type $B$); hence $(K,S)$ is of type 11. It is impossible for $C$ and $C'$ both to contain two points - for if this were so then each of $C$ and $C'$ would be $S$-adjacent both to $B$ and to $B'$.
Therefore one of $C$ and $C'$ ($C$ say) contains just one point; similarly one of $B$ and $B'$ ($B$ say) contains just one point. Let $x$ denote the unique point in $C$, and let $L$ denote the set of $S$-neighbours of $x$ in $K$. We claim that all three points in $L$ must belong to the same $A$-component of $S^0K$. For if $u$ and $v$ are any two distinct points in $L$ then there is a point $y$ in $S\setminus\{x\}$ which is a $S$-neighbour of both $u$ and $v$; so since
adj(A,K,S) is acyclic and \{x\} is an A-component of S\(\Box\)K it follows that u and v
belong to the same A-component of S\(\Box\)K, as we claimed. Hence L = B'. Similarly, if
y is the unique point in B and M is the set of 6-neighbours of y in K then M = C'.
It follows that K is as shown in Figure 2.

(iv) Let C be an A\(\Box\)X-component of S\(\Box\)X and let B be an A\(\Box\)X-component of S\(\Box\)\(\Box\)X that is
A\(\Box\)X-adjacent to C. Let K be a unit cell in X such that C\(\Box\)K is A\(\Box\)X-adjacent to
B\(\Box\)K. If K is as in Figure 2 then the result is plainly true. If K is not as in Figure 2
then by (iii) either C\(\Box\)K = S\(\Box\)K or B\(\Box\)K = S\(\Box\)\(\Box\)K; but B\(\Box\)K and C\(\Box\)K are certainly 6-adjacent
to S\(\Box\)K and S\(\Box\)\(\Box\)K respectively (since K\(\Box\)\(\Box\)\(\Box\) is 6-connected), and so we are home.

(v) Let D denote the (A, S\(\Box\))-border of S. Then S\(\Box\) is a union of A-components of B\(\Box\), so
the result follows from (i) and (ii). □

Note that the converse of Proposition 1 (v) is false; for if A is the 6-adjacency relation
and \(S = \{ (x,y,z) \mid |x|+|y|+|z| \leq 1 \}\) then S is A-normal but the (A, S\(\Box\))-border of S is
not.

The following theorem, which is our principal result, establishes a fundamental global
property of three-dimensional binary digital pictures.

DEFINITION

When d = 3, we say that a digital picture (A, S) is strongly normal if adj(A,K,S) is a
tree for all unit cells K, and, for every face F of a unit cell either F\(\Box\)S or F\(\Box\)\(\Box\)S is
A-connected. When d = 2 we say that (A, S) is strongly normal if, for all unit cells K
such that K\(\Box\)S consists of two diagonally opposite corners of K, exactly one of K\(\Box\)S
and K\(\Box\)\(\Box\)S is A-connected. □
Diagram to Accompany Proposition 1(iii)

FIGURE 2:  A - First A-component of $S \triangleleft K$
            B - First A-component of $S^c \triangleleft K$
            C - Second A-component of $S \triangleleft K$
            D - Second A-component of $S^c \triangleleft K$
THEOREM 2
Suppose \((A,S)\) is a three-dimensional binary digital picture. Then the following are equivalent:
I. \(\text{adj}(A,X,S)\) is a tree for all simply-connected windows \(X\).
II. \((A,S)\) is normal.
III. \((A,S)\) has a continuous analogue relative to every window.
IV. \((A,S)\) has a continuous analogue relative to every unit cell.

PROOF
I implies II; III implies IV
These implications are trivial.

II implies III
Let \(A'\) be the adjacency relation such that \(x\) is \(A'\)-adjacent to \(y\) if and only if either \(x\) and \(y\) are \(A\)-adjacent, or there is a unit cell \(K\) that contains \((x,y)\) and \(x\) and \(y\) are in the same \(A\)-component of \(SMK\).

Suppose \((A,S)\) is normal. Then \((A',S)\) is strongly normal. So by Proposition 1 in the next chapter \((A',S)\) has a continuous analogue \(C^*(A',S)\). Plainly \(C^*(A',S)\) will also be a continuous analogue of \((A,S)\).

IV implies II; III implies I
Suppose \((A,S)\) has a continuous analogue relative to a simply-connected window \(X\) (thus \(X\) might be a unit cell or the whole of \(R^3\)). We shall show that \(\text{adj}(A,X,S)\) is a tree.
Suppose (for the purpose of getting a contradiction) that \(\text{adj}(A,X,S)\) contains a cycle.
Let \(U\) and \(V\) be distinct \(A\)-components of \(SMX\) which correspond to two vertices on the cycle. Pick a point \(u\) in \(U\) and a point \(v\) in \(V\).

By definition of \(u\) and \(v\) we may partition \(SMX\) into two subsets \(M\) and \(N\) such that each of \(M\) and \(N\) is a union of \(A\)-components of \(SMX\) and such that there exist \(A\)-paths \(P_0\) in \(XM^C\)
and $Q_0$ in $X\cap N$ each of which links $u$ to $v$. Now let $C$ be a continuous analogue of $(A,S)$ relative to $X$. By Proposition 0(i) there exist closed sets $F$ and $B$, each of which is a union of connected components of $C$, such that $F \cap Z^2 = M$ and $B \cap Z^2 = N$. By conditions (ii), (iii) and (v) of the definition of a continuous analogue $u$ and $v$ are connected in $X \setminus F$ and in $X \setminus B$. $X \setminus F$ and $X \setminus B$ are both open relative to $X$, so there exist two curves $\xi_0$ and $\xi_1$ in $X$ joining $u$ to $v$ such that the trace of $\xi_0$ does not meet $B$ and the trace of $\xi_1$ does not meet $F$.

By our definition of simple-connectedness there exists a continuous map $h : [0,1] \times [0,1] \to X$ such that for all $s$ and $t$ in $[0,1]$: 

(i) $h(s,0) = \xi_0(s)$

(ii) $h(s,1) = \xi_1(s)$

(iii) $h(0,t) = u$

(iv) $h(1,t) = v$

In the rest of this proof the term 'square' denotes a closed two-dimensional square (and not just the interior or boundary of the 'square'). By 'drawing' lines parallel to the $x$ and $y$ axes we may dissect the unit square $[0,1] \times [0,1]$ into $n^2$ little squares of side $1/n$, where $n$ is chosen so large that the image under $h$ of each little square always misses at least one of the two sets $F$ and $B$. Let $Q$ be the set of all the little squares whose image under $h$ meets $F$. Define $E_0 = \{ e \mid e$ is an edge of exactly one square in $Q \}$. Let $E_1$ denote the straight line segment with endpoints $(0,0)$ and $(1,0)$ and let $E_2$ denote the set $\{ e \mid e$ is an edge of a little square and $e \in L \}$. Let $E$ denote the symmetric difference of $E_0$ and $E_1$. With the exceptions of $(0,0)$ and $(1,0)$ each corner of a little square is incident either on no edges in $E$ or on exactly two edges in $E$. So since each of $(0,0)$ and $(0,1)$ is incident on exactly one edge in $E$ it follows that $(0,0)$ and $(0,1)$ belong to the same connected component of $UE_2$; call this connected component $P$. Then $h(P)$ is a connected subset of $X$ which contains both $u$ and $v$ but does not meet $F$ or $B$.

So since $X \setminus C$ is open in $X$ there exists a simple polygonal arc in $X \setminus C$ which joins $u$ and $v$. So by Proposition 0 $u$ and $v$ belong to the same $A$-component of $S^2 \setminus X$, which is the required
contradiction. ■

Theorem 2 implies that if a closed set has a digital representation then all its digital representations are normal. Another easy corollary is the following generalization of Corollary 7 in [Rosenfeld81].

COROLLARY
Suppose one of α and β is not equal to 6. Then adj(α, β, s) is a tree for every s.

PROOF
We have already noted that (α, β, 6, s) is normal whenever α and β are not both equal to 6; so the Corollary follows from the Theorem. ■

REMARK
Theorem 2 remains true if property I is replaced by the more general assertion:

I' For all windows X, the Euler characteristic of adj(A, X, s) is not less than the Euler characteristic of X minus the number of cavities of X.

(This says that adj(A, X, s) never has more 'holes' than X. Thus if X has just one hole then adj(A, X, s) has at most one cycle.) ■

Proposition 1 (ii) implies that if (A, W) is normal then every A-component of W is A-normal. The converse of this result is false; indeed, if A is the 6-adjacency relation and $S = \{(x, y, z) \mid |x| + |y| + |z| = 1 \}$ then adj(A, S) is the complete bipartite graph $K(6, 2)$ which is not a tree. However, a partial converse is provided by the corollary to the following proposition.

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PROPOSITION 3
Let \( X \) be a window, and let \((A,T)\) be a binary digital picture such that \( \text{adj}(A,X,T) \) is a tree. Let \( \{ T_i \mid 0 \leq i \leq n \} \) be the set of \( A(X) \)-components of \( T^{\text{in}}X \) and let \( W \) be a subset of \( T \) such that for each \( i \) the adjacency graph \( \text{adj}(A,X,T_i,W) \) is a tree. Then \( \text{adj}(A,X,W) \) is a tree.

PROOF
Suppose the hypotheses are satisfied but \( \text{adj}(A,X,W) \) contains a cycle \( \Gamma \). Let \( W_i \) denote \( T_i^{\text{in}}X \). We shall deduce the contradiction that for some \( i \) \( \text{adj}(A,X,W_i) \) contains a cycle.
Let \( a, x \), and \( b \) be three consecutive vertices on \( \Gamma \) such that \( x \) is a \( W \)-vertex, and let \( T_0 \) be the \( A(X) \)-component of \( T^{\text{in}}X \) that contains \( \text{COMPT}(x) \). Then \( \text{COMPT}(x) \) is an \( A(X) \)-component of \( W_0^{\text{in}}X \) while \( \text{COMPT}(a) \) and \( \text{COMPT}(b) \) are \( A(X) \)-components of \( W_0^{\text{in}}X \). It follows that \( \text{COMPT}(x) \) is a union of \( A(X) \)-components of \( W_0^{\text{in}}X \), and that each of \( \text{COMPT}(a) \) and \( \text{COMPT}(b) \) is contained in an \( A(X) \)-component of \( W_0^{\text{in}}X \). By their definition \( \text{COMPT}(a) \) and \( \text{COMPT}(b) \) are both \( A(X) \)-adjacent to \( \text{COMPT}(x) \), but since \( \Gamma \) is a cycle they are not separated by \( \text{COMPT}(x) \) (i.e. there exists an \( A(X) \)-path in \( \text{COMPT}(x)^{\text{in}}X \) from \( \text{COMPT}(a) \) to \( \text{COMPT}(b) \)). It is a hypothesis of this proposition that \( \text{adj}(A,X,W_i) \) is a tree; hence the last sentence implies that \( \text{COMPT}(a) \) and \( \text{COMPT}(b) \) are contained in the same \( A(X) \)-component of \( W_0^{\text{in}}X \).

Pick \( u \) and \( v \) on the \( (A(X),\text{COMPT}(x)) \)-borders of \( \text{COMPT}(a) \) and \( \text{COMPT}(b) \) respectively. By the last sentence in the previous paragraph there exists an \( A(X) \)-path \( P \) in \( X \) which joins \( u \) to \( v \) and which does not meet \( W_0 \). Now \( a \neq b \) means \( \text{COMPT}(a) \) and \( \text{COMPT}(b) \) are different \( A(X) \)-components of \( W_0^{\text{in}}X \). Hence \( P \) contains at least one point in \( W \) and since \( P \) does not meet \( W_0 = T_0^{\text{in}}W \) it follows that every point \( w \) in \( W \) that lies on \( P \) belongs to \( (T \setminus T_0) \). For every point \( u \) in \( W \) that lies on \( P \) let \( z(u) \) and \( z'(u) \) denote the two points on \( P \) which are such that: (a) \( z(u) \) and \( z'(u) \) both lie in \( T_0 \); (b) \( z(u) \) comes before \( w \) and \( z'(u) \) comes after \( w \) on \( P \); (c) the portion of \( P \) between \( z(u) \) and \( z'(u) \) contains no other points in \( T_0 \). Also, let \( r(u) \) denote the immediate successor of \( z(u) \) on \( P \) and let \( s(u) \) denote the
immediate predecessor of \( z'(w) \) on \( P \). Suppose temporarily that there is no point \( w \) such that \( r(w) \) and \( s(w) \) belong to different \( A(X) \)-components of \( T^D_X \). Then we can construct an \( A(X) \)-path \( P' \) in \( (T_0 \setminus W) \cup (T^D_X) \) that joins \( u \) to \( v \). \( P' \) passes through each \( r(w) \) and \( s(w) \) but bypasses every point \( w \) in \( W \) that lies on \( P \). The \( A(X) \)-path \( P' \) does not meet \( W \), which implies that \( a = b \). This contradiction shows that there must exist a point \( w_0 \) such that \( r(w_0) \) and \( s(w_0) \) belong to different \( A \)-components of \( T^D \). But by construction \( r(w_0) \) and \( s(w_0) \) are both \( A(X) \)-adjacent to \( T_0 \), and they are not separated by \( T_0 \). Hence \( \text{adj}(A,X,T) \) contains a cycle and this contradiction proves the Proposition. \( \blacksquare \)

**COROLLARY**

Let \( T \) be an \( A \)-normal set and let \( W \) be a subset of \( T \) which meets each \( A \)-component of \( T \) in an \( A \)-normal set. Then \( W \) is \( A \)-normal.

**PROOF**

Let \( K \) be an arbitrary unit cell and let \( T_0 \) be any \( A \)-component of \( T^D_K \). Let \( T'_0 \) be the \( A \)-component of \( T \) that contains \( T_0 \). Then \( \text{adj}(A,K,T'_0,W) \) is a tree, so by Proposition 1 (i) \( \text{adj}(A,K,T'_0,W) \) is a tree. Also, \( \text{adj}(A,K,T) \) is a tree since \( T \) is \( A \)-normal. So since \( T_0 \) was any \( A \)-component of \( T^D_K \), Proposition 3 implies that \( \text{adj}(A,K,W) \) is a tree. This argument works for all unit cells \( K \), so \( W \) is \( A \)-normal. \( \blacksquare \)

We are now ready to prove what might well be regarded as the fundamental connectivity and separation properties of digital borders. Basically, we give conditions which ensure that the border of a component of object points with respect to a component of background points is connected and separates the components. If any digital picture contains a border which does not have these two properties then it is clear that the adjacency relation is incompatible with the set of object points.
PROPOSITION 4

Let F and B be A-components of S and S^c respectively and let W denote the (A,B)-boder of F. Suppose both S and W are A-normal. Then W is A-connected. Furthermore, if W is non-empty and P is an A-path linking a point in F to a point in B then there are two consecutive points u, v on P such that u ∈ W and v ∈ B.

PROOF

Let T denote the set W∪(S\F).

Suppose W is not A-connected. Let U and V be distinct A-components of W. Then there exists an A-path in F that links U to V; on such an A-path pick a point x that does not belong to W, and let X denote the A-component of T^c that contains x. U and V are distinct A-components of T. Furthermore, B is an A-component of T^c and B ≠ X. Now there exist two A-paths P0 and P1 from x to B such that P0 goes through V and does not meet U while P1 goes through U and does not meet V. It follows that there exist two different paths in adj(A,T) from X to B (of which one contains U and the other contains V). But R^3 is a simply-connected window; therefore Theorem 2 implies that T is not A-normal.

But, by the Corollary to Proposition 3, T is A-normal since T meets each A-component of S in an A-normal set. This contradiction proves the first assertion.

To prove the second assertion suppose W is non-empty, and let z be any point in F. If z ∈ W define C = W; if z ∈ F\W then let C be the A-component of T^c that contains z. Let P be an A-path from z to a point in B. Let b, c and w be the vertices of adj(A,T) such that COMPT(b) = B, COMPT(c) = C and COMPT(w) = W, so that w is adjacent to b and either w = c or else w is adjacent to c. We have already noted that T is A-normal, so adj(A,T) is a tree. Now the A-path P induces a walk in adj(A,T) from c to b, and this walk must traverse the edge in adj(A,T) joining w to b (since adj(A,T) is acyclic). This implies the result. ■
The next proposition is the analogue of our Proposition 4 in the conventional theory of binary digital pictures. The proposition contains Proposition 5 in [Rosenfeld81] as a special case and can also be shown to imply the principal theorem in [ArtzyFriederHereman81] (use the technique described in the final section of [MorgensternRosenfeld81]). So by proving it we achieve the second of our goals. Results of this kind are used to establish the soundness of border-tracking algorithms.

**PROPOSITION 5**

Let each of $\alpha$, $\beta$ and $\delta$ be equal to 6, 18 or 26, and let $\delta = \min(\beta, \delta)$. Let $F$ be an $\alpha$-component of $S$. Let $A$ be a $\beta$-component of $S^C$ and let $W$ denote the $(\delta, \beta)$-border of $F$.

(i) Suppose at least one of $\alpha$ and $\delta$ is not equal to $\beta$. Then $W$ is $\alpha$-connected.

(ii) Suppose $W$ is non-empty and at least one of $\alpha$ and $\beta$ is not equal to $\delta$. Then any $\delta$-path $P$ from a point in $F$ to a point in $B$ contains two consecutive points $u$, $v$ such that $u \in W$ and $v \in B$.

**PROOF**

**Case 1:** $\alpha$ and $\delta$ are not both equal to $\beta$.

We shall prove (i) and (ii) simultaneously.

Let $A$ denote the adjacency relation on $Z^2$ such that $x$ and $y$ are $A$-adjacent either if they are $(\alpha, \beta, \delta, 5)$-adjacent or if $x$ and $y$ are $26$-neighbours in $S \setminus W$. Observe that $B$ is an $A$-component of $S^C$.

Let $K$ be any unit cell. If $\alpha = 6$ (so that neither $\beta$ nor $\delta$ is equal to $6$) then either $S^C K$ is $6$-connected (and hence $A$-connected) or else $S^C K$ is $18$-connected (and hence $A$-connected). If on the other hand $\alpha \neq 6$ then either $S^C K$ is $18$-connected (and hence $A$-connected).
A-connected) or else $S^{cnk}$ is $G$-connected (and hence $A$-connected). Thus $S$ is $A$-normal in all cases.

Let $F'$ denote the $A$-component of $S$ that contains $F$, and let $W'$ denote the $(A,B)$-border of $F'$ (so $W'$ is the $(A,B)$-border of $F'$). Let $K$ be any unit cell. If $\alpha = 5$ (so that neither $B$ nor $\delta$ is equal to 6) then either $W^{nk}$ is $G$-connected (and hence $A$-connected) or else $W^{cnk}$ is $18$-connected (and hence $A$-connected, since $SN'\subseteq SW$). If on the other hand $\alpha \neq 5$ then either $W^{nk}$ is $18$-connected (and hence $A$-connected) or else $W^{cnk}$ is $6$-connected (and hence $A$-connected). Thus $W'$ is $A$-normal in all cases. Recalling that $S$ is also $A$-normal, we see that by Proposition 4 the set $W'$ is $A$-connected. But it is easy to see that $W = W^{NF}$ so since $F$ is an $\alpha$-component of $S$, $W$ is not $\alpha$-adjacent to $W'\setminus W$. Therefore $W$ is a subset of $W'$ that is not $\alpha$-adjacent to $W'\setminus W$. But both $W$ and $W'$ are $A$-connected; hence $W = W'$. This proves part (i) of the Proposition. Since $W$ is $A$-connected it follows that given any $\delta$-path $P$ we can construct an $A$-path $P'$ which agrees with $P$ except possibly on $W$ (in the sense that if $x$ and $y$ are consecutive points on $P$ which are not both in $W$ then $x$ and $y$ are consecutive points on $P'$, and vice versa). Thus we are home by Proposition 4 (applied to $A$, $F'$, $B$ and $W' = W$).

**Case II: $\alpha = 5$ and $\delta = 6$**

Here only part (ii) of the Proposition applies. Note that $B \neq 5$ (by the hypothesis that at least one of $\alpha$ and $\delta$ is not equal to 6). Let $P$ be a $\delta$-path (i.e. a $6$-path) from a point in $F$ to a point in $B$. Let $\delta' = 26$, let $W''$ be the $(\delta',B)$-border of $F$ and let $\delta' = \min(\delta,\delta') = 8$. Then since the $6$-path $P$ is (a fortiori) a $\delta'$-path, and since $\delta' \neq 5$, it follows from Case I (when $\delta$ and $\delta'$ are replaced by $\delta'$ and $\delta'$) that there are two consecutive points $u$, $v$ on $P$ such that $u \in W''$ and $v \in B$. But $u \in W''$ implies $u \in F$, and since $u$ and $v$ are consecutive on $P$, $u$ is $5$-adjacent to the point $v$, which is in $B$. Hence $u \in W$, and we are home. 

We can show that Proposition 5 is a 'best possible' result. First of all, it is obvious
that $W$ will not in general be more than $\alpha$-connected, simply because that is the only
sort of connectivity that is assumed for $F$. Now put $S = XUYU((-2,0,0),(-2,0,0))$ where $X$
$= \{(x,y,z) \in \mathbb{Z}^3 \mid \max(|x|,|y|,|z|) = 2 \text{ and just one of } |x|, |y|, |z| \text{ is equal to } 2\}$, and
$Y = \{(x,y,z) \in \mathbb{Z}^3 \mid |x|+|y|+|z|\geq 3\}$. We shall consider the consequences of different
choices of $\alpha$, $\beta$ and $\delta$. In all cases let $B$ denote the only unbounded $\beta$-component of
$S^c$, let $z_0$ be the point $(0,0,0)$, let $F$ denote the $\alpha$-component of $S$ that contains $z_0$,
and let $W$ denote the $(\delta,B)$-border of $F$. If $\alpha = \delta = 5$ then $W$ is not $\alpha$-connected, so part
(i) fails; if $\alpha = \beta = 6$ then $z_0$ is contained in the same $\delta$-component of $W^c$ as $B$ so part (ii)
fails. Also, part (ii) cannot be strengthened by substituting $\beta$ or $\delta$ for $\delta$: if $\alpha = \beta = 18$
and $\delta = 6$ then $z_0$ is contained in the same $\beta$-component of $W^c$ as $B$ while if $\alpha = \delta = 18$ and
$\beta = 6$ then $z_0$ is contained in the same $\delta$-component of $W^c$ as $B$. Even if $\delta = 18$, part (ii)
may fail if $\delta$ is replaced by $\beta$ or $\delta$. To see this put $(\beta, \delta) = (25, 18)$ or $(18, 25)$ and $S$
$= F = X \setminus Y$ where $X = \{(x,y,z) \in \mathbb{Z}^3 \mid \max(|x|,|y|,|z|)\geq 2\}$ and $Y = \{(x,y,z) \in \mathbb{Z}^3 \mid |x|=|y|=|z|\geq 2\}$.

Applications to Other Kinds of Digital Pictures

The Two-Dimensional Case

It is easily seen that Proposition 1 remains true in two dimensions — but in part (iv)
'6-adjacent' should be replaced by '4-adjacent', and the assumption that $(A,S)$ is normal
is unnecessary.

Unfortunately Theorem 2 fails when $(A,S)$ is a two-dimensional binary digital picture;
indeed, if $S = \{(x,y) \in \mathbb{Z}^2 \mid |x|+|y| = 1\}$ and $A$ is the 6-adjacency relation then $(A,S)$ is
normal but has no continuous analogue. (The fact that $(A,S)$ has no continuous analogue is
the essence of the 'paradox' mentioned in [RosenfeldPfaltz55], p175). If $S$ is as above
and $A$ is the 4-adjacency relation then $(A,S)$ again has no continuous analogue — this can
be regarded as the source of the 'Euler Theorem paradox' mentioned in ([Rosenfeld70],
p147), where these paradoxes are put forward as a reason for not using the same type of
connectivity for both set and complement'. But note that in the second case \((A,S)\) is not normal, and so would not be expected to have a continuous analogue.) It is easy to prove the following weak analogue of Theorem 2:

**THEOREM 2'**

Suppose \((A,S)\) is a two-dimensional binary digital picture. Then \(\text{adj}(A,X,S)\) is a tree for all simply-connected windows \(X\) if and only if \((A,S)\) is normal.

**PROOF**

Let \((A,S)\) be a two-dimensional binary digital picture. Define an adjacency relation \(A'\) on \(\mathbb{Z}^2\) such that two points \((x,y,z)\) and \((x',y',z')\) are \(A'\)-adjacent iff they are 26-adjacent and the points \((x,y)\) and \((x',y')\) are either equal or \(A\)-adjacent. Define \(S'\) to be the set \(S\times[0,1]\). Then if \(X\) is any two-dimensional window we have that \(\text{adj}(A,X,S)\) is a tree if and only if \(\text{adj}(A',X\times[0,1],S')\) is a tree (here \([0,1]\) denotes the closed unit interval \(\{x \mid 0 \leq x \leq 1\}\)). Hence I and II are equivalent by Theorem 2.

**REMARK**

In this chapter we never make use of the assumption that an \(n\)-dimensional window is actually embedded in \(n\)-dimensional Euclidean space. Thus one might define a generalized window to be any space obtainable by taking a disjoint collection of unit cells and 'gluing together' some of the corners, edges and (in three dimensions) faces of these cells. Observe that any union of faces of three-dimensional unit cells is a two-dimensional generalized window, regardless of whether or not the faces all lie in a plane. Theorems 2 and 2' remain valid if \(X\) is a generalized window.

Propositions 3 and 4 remain true in two dimensions: the proofs are obtained from the three-dimensional proofs by replacing references to Theorem 2 by references to Theorem 2'. If in the statement and proof of Proposition 5 we replace '26' and '18' by '8', and we replace '6' by '4' then we get a correct statement and proof of a two-dimensional version of Proposition 5. This is again a 'best possible' result - two-dimensional versions of the
The Hexagonal and Face-Centred Cubic Lattices

It is unfortunate that, in general, neither of the two-dimensional binary digital pictures (4,5) and (8,5) has a continuous analogue relative to any given window; the same applies to the three-dimensional picture (6,5). As a result, digital objects in these pictures can have properties which real objects never have—recall the counterexamples given after the proof of Proposition 5. Although continuous analogues always exist for the binary digital pictures (18,5) and (26,5) (because they are always normal) these continuous analogues can have rather counterintuitive topological properties. As an example, consider the digital picture \((0,\{(0,0),(1,1),(0,1)\})\) where \(c = 18\) or \(26\): although one feels that this digital picture ought to be simply connected, since any two object points are mutually adjacent, it does not have a simply connected continuous analogue (relative to \(\mathbb{R}^3\)). A related problem is that if we define the Euler characteristic of (18,5) and (26,5) to be the Euler characteristic of the natural continuous analogues of these pictures, then the topological structure of a picture may change when a simple point (precisely defined in the next chapter) is removed. A more obvious possible drawback of (18,5) and (26,5) is that the 18- and 26-adjacency relations give each point too many neighbours; it is desirable to have fewer neighbours, as this will reduce the amount of computation involved in many algorithms.

The standard way of avoiding these difficulties is to use adjacency relations of the form \((\alpha, \beta, \beta, 5)\), where 5 is the set of object points, and exactly one of \(\alpha\) and \(\beta\) (usually \(\beta\)) is equal to 4 or 6. We shall consider an alternative approach.

In the two-dimensional case the alternative we are about to describe is in fact quite well-known (e.g. (Goley69, Hilditch69, Deutsch72)), although it has never been widely used in practice. It involves abandoning the rectangular grid in favour of the hexagonal lattice which is constructed by tessellating the plane with unit equilateral triangles and regarding each point lying at a corner of a triangle as a lattice point. The hexagonal
lattice is so called because it is possible to tessellate the plane with regular hexagons in such a way that the set of lattice points is exactly the set of centres of the hexagons.

The only adjacency relation we shall use with this lattice is the relation $A$ such that each lattice point is $A$-adjacent to its six nearest neighbours and to no other points. A binary digital picture on the hexagonal lattice is just a set of lattice points (since the adjacency relation is fixed there is no need to include it in the definition). The relation $A$ assigns just 6 neighbours to each lattice point, which is an improvement on the 8-adjacency relation on the rectangular lattice. Actually, the hexagonal lattice has another advantage over 8-adjacency; in the hexagonal lattice each lattice point is equidistant from all its neighbours. More generally, the adjacency relation $A$ is isotropic in the sense that if $p$, $q$ and $r$ are any three lattice points in the hexagonal lattice such that $p$ and $q$ are both adjacent to $r$ then there is a plane rotation with centre $r$ which preserves the lattice and maps $p$ to $q$.

Digital pictures on the hexagonal lattice can be incorporated into the theory of digital pictures developed above. Indeed, if $A'$ denotes the adjacency relation on the rectangular lattice $Z^2$ such that $(x,y)$ is $A'$-adjacent to $(s,t)$ iff $(x,y)$ is $4$-adjacent to $(s,t)$ or $(x-s,y-t)\in\{(1,1),(-1,-1)\}$, then it is well-known (and in any case easy to see) that there is an affine map $T$ of the plane to itself which induces a bijection of the points of the hexagonal lattice to $Z^2$ with the property that two lattice points in the hexagonal lattice are $A$-adjacent iff their images in the rectangular lattice are $A'$-adjacent. So for the purposes of digital topology every binary digital picture $W$ on the hexagonal lattice is equivalent to the binary digital picture $(A',T(W))$ on the rectangular lattice.

It is readily confirmed that $(A',S)$ is normal for all $S$. Moreover, a very well-behaved continuous analogue for $(A',S)$ relative to an arbitrary window $X$ can be constructed by taking the union of all points in $S^0X$, all straight line segments joining two $A'$-adjacent points in $S^0X$, and all $(1,1,\sqrt{2})$ triangles each of whose sides joins two $A'$-adjacent points.
in $\mathbb{Z}^n$. Thus the "borders are connected and surround" result (Proposition 1) holds for digital pictures on a hexagonal lattice. Note that this is a property of the hexagonal lattice itself, and not merely a property of $A'$.

* The three-dimensional analogue of the two-dimensional hexagonal lattice is the face-centred cubic lattice whose set of lattice points can be taken to be $\{(x,y,z) \in \mathbb{Z}^3 | x+y+z \text{ is even}\}$. The only adjacency relation we shall use with this lattice is the relation $A_1'$ in which each lattice point is adjacent to its nearest neighbours. As the adjacency relation is again fixed we can define a binary digital picture on the lattice to be any set of lattice points. The adjacency relation $A_1$ assigns 12 neighbours to each lattice point, which is an improvement on the 16- or 26-adjacency relations on the rectangular lattice. Like the adjacency relation $A$ on the two-dimensional hexagonal lattice the adjacency relation $A_1'$ is isotropic, which the 16- and 26-adjacency relations are not.

It is interesting to note that there can tessellate three-space with equal rhombic dodecahedra in such a way that the set of centres of the rhombic dodecahedra is exactly the points set of the face-centred cubic lattice. (The convex polyhedron whose vertex set is $\{(x,y,z) \in \mathbb{Z}^3 | (|x|,|y|,|z|) = (1,1,1) \text{ or } (2,0,0) \text{ or } (0,2,0) \text{ or } (0,0,2)\}$ is a rhombic dodecahedron.)

As in the case of the hexagonal lattice, digital pictures on the face-centred lattice can be incorporated into our general theory. There are two natural ways of doing this. The first way is based on the adjacency relation $A_1'$ on the rectangular grid such that the point $(x,y,z)$ is $A_1'$-adjacent to the point $(t,u,v)$ if and only if the two points are 6-adjacent, or $(x-t,y-u,z-v) \in \{(1,0,1),(0,1,1),(-1,1,0),(-1,0,-1),(0,-1,-1),(1,-1,0)\}$. The other way is based on the adjacency relation $A_1''$ on the rectangular grid such that the point $(x,y,z)$ is $A_1''$-adjacent to the point $(t,u,v)$ if and only if the two points are 6-adjacent, or $(x-t,y-u,z-v) \in \{(1,0,1),(0,1,1),(1,1,1),(-1,0,-1),(0,-1,-1),(-1,1,1)\}$. There are affine transformations $T_1'$ and $T_1''$ which map the points of a face-centred cubic.
lattice onto the points of a rectangular lattice in such a way that two points of the face-centred cubic lattice are \( A_1 \)-adjacent iff their images under \( T_1 \) and \( T_1' \) are respectively \( A_1' \)-adjacent and \( A_1'' \)-adjacent. So for the purposes of digital topology every binary digital picture on the face-centred cubic lattice is equivalent to binary digital pictures on the rectangular lattice based on the adjacency relations \( A_1' \) and \( A_1'' \).

We claim that \((A_1', S)\) and \((A_1'', S)\) are normal for all \( S \). To justify this claim, it suffices to show that each of \( \text{adj}(A_1', K, S) \) and \( \text{adj}(A_1'', K, S) \) is a tree when \( K \) is the unit cell that contains \((0,0,0)\) and \((1,1,1)\). We may assume WLOG (by symmetry between \( S \) and \( S' \)) that \((0,0,0) \in S\). If \((1,0,0)\) or \((0,1,0)\) is also in \( S \) then \( S'K \) is \( A_1' \)-connected so \( \text{adj}(A_1', K, S) \) is a tree. If both of these points are in \( S' \) then the set consisting of these two points (which are contained in one \( A_1' \)-component of \( S'\bar{N}K \)) is \( A_1' \)-adjacent to every \( A_1' \)-component of \( S'\bar{N}K \). Moreover, if there is a second \( A_1'' \)-component of \( S'\bar{N}K \) than it must be \((0,0,1)\) (since all other points in \( S'\bar{N}K \) are \( A_1' \)-adjacent to \((1,0,0) \) or \((0,1,0))\), and \((0,0,1)\) is adjacent to only one \( A_1'' \)-component of \( S'\bar{N}K \). So \( \text{adj}(A_1'', K, S) \) is a tree in all cases. Again, if one of the three points \((1,0,0), (0,1,0) \) and \((1,1,1) \) is in \( S \) than \( S'K \) is \( A_1'' \)-connected, whence \( \text{adj}(A_1'', K, S) \) is a tree. If all three of these points are in \( S' \) then the set consisting of these three points (which are contained in one \( A_1'' \)-component of \( S'\bar{N}K \)) is \( A_1'' \)-adjacent to every \( A_1'' \)-component of \( S'\bar{N}K \). Moreover, if there is a second \( A_1'' \)-component of \( S'\bar{N}K \) than it must be \((0,0,1)\), which is \( A_1'' \)-adjacent to only one \( A_1'' \)-component of \( S'\bar{N}K \). So \( \text{adj}(A_1'', K, S) \) is a tree in all cases.

It follows that the 'borders are connected and surround' result (Proposition 4) holds for digital pictures based on the face-centred cubic lattice. Although this result was proved by consideration of \( A_1' \) (or \( A_1'' \)), it expresses a property of the face-centred cubic lattice itself.

*The principal ideas in the rest of this section are due to Dr. Rocco*.
Summary

This chapter may be summarized as follows:

(a) We introduced a new definition of a *binary digital picture* (based on the lattice point representation of voxels); the new definition was more general but arguably rather simpler than the conventional definition.

(b) We adopted the adjacency graphs of [Buneman59] and [Rosenfeld74] to the new definition, and at the same time introduced the slightly broader concept of the \( X \)-adjacency graph of a binary digital picture, where \( X \) can be any window (i.e. any union of unit cells).

(c) We defined a *normal* binary digital picture to be a picture \((A, S)\) whose \(K\)-adjacency graph is a tree for every unit cell \(K\).

(d) We introduced the notion of the *continuous analogue* of a binary digital picture relative to a window.

(e) Using the method of continuous analogues, we proved satisfying theorems (Theorems 2 and 2'), which show that the concept of normality is of fundamental importance in the theory of binary digital pictures.

(f) Fundamental separation and connectivity properties of digital borders were deduced from Theorems 2 and 2' (without making further use of the powerful but rather complex machinery of continuous analogues). These results are general forms of Proposition 5 in [Rosenfeld81] and they are useful for proving the soundness of border tracking algorithms.

(g) We showed how binary digital pictures on the hexagonal and face-centred cubic lattices with nearest neighbour adjacency could be incorporated into the theory, and explained
why they are well-behaved.

Our approach to digital topology (based on the new definition of a binary digital picture) produces stronger theorems than the conventional approach. Thus no result as powerful as Theorem 2 could be stated in terms of the old theory, and so it would be impossible to prove a result like Proposition 5 in the way we have. The relationship of our approach to previous ones is akin to that between the study of topology and the study of particular topological spaces. The generality of our approach has paid off in the applications (f) and (g) above which have emerged. Moreover, the new theory can cope with a greater variety of different adjacency relations than the old theory and this increased power might have applications in situations where our pictorial data is 'noisy' and of low resolution.

Appendix to Chapter 2

A. Simple-Connectedness

In most texts (e.g. [Apastol74]) a connected set Y is said to be simply-connected iff every closed curve in Y can be continuously deformed to a single point. Thus Y is simply connected iff given any curve \( \gamma : [0,1] \to Y \) such that \( \gamma(0) = \gamma(1) \) there exists a continuous function \( h : [0,1] \times [0,1] \to Y \) with the following properties:

(i) \( h(x,0) = \delta(x) \) \( (0 \leq x \leq 1) \)

(ii) \( h(x,1) = p \) \( (0 \leq x \leq 1) \) where \( p \) is a point in Y and is the same for all \( x \)

(iii) \( h(0,t) = h(1,t) \) \( (0 \leq t \leq 1) \)

We shall call this the standard definition.

If Y is simply-connected according to our earlier definition then it is readily seen to be simply-connected in the standard sense. To prove the converse suppose Y satisfies the standard definition and suppose \( \delta_0 \) and \( \delta_1 \) are two curves on Y such that \( \delta_0(0) = \delta_1(0) \) and
\[ x_0(1) = x_1(1). \] We must prove the existence of a fixed endpoint homotopy \( H \) of \( \delta_0 \) onto \( \delta_1 \).

Define a closed curve \( \kappa \) such that \( \kappa(x) = \kappa_0(2x) \) if \( x \in [0, 1/2] \) and \( \kappa(x) = \kappa_1(2(1-x)) \) if \( x \in [1/2, 1] \). By hypothesis there exists a continuous function \( h \) satisfying (i), (ii) and (iii) above. Now define \( g_0, g_1, G_0, G_1, H : [0, 1] \times [0, 1] \to Y \) such that:

\[ g_0(x, t) = h(x/t, 2t) \quad g_1(t) = h(1-x/t, 2t); \]

for \( j = 0, 1 \)

\[ G_0(x, t) = g_j(0, 2t) \text{ if } x < t/2, \]

\[ G_1(x, t) = g_j(1, 2(1-t)) \text{ if } x > 1-t/2, \]

\[ G_j(x, t) = g_j((2x-t)/(2(1-t)), t) \text{ otherwise; } \]

\[ H(x, t) = G_0(x, 2t) \text{ if } t \leq 1/2, \]

\[ H(x, t) = G_1(x, 2(1-t)) \text{ if } t \geq 1/2. \]

Then \( H(x, 0) = \delta_0(x) \);

\[ H(x, 1) = \delta_1(x); \]

\[ H(0, t) = \delta_0(0) = \delta_1(0); \]

\[ H(1, t) = \delta_0(1) = \delta_1(1). \]

It is easily verified that \( H \) is continuous. (The only point where there is any doubt is \((1/2, 1)\), which is a discontinuity of \((2x-t)/(2(1-t))\); however, all is well because \( g_j \) is uniformly continuous \((j = 0, 1)\); given any \( \epsilon > 0 \) we can pick \( \delta > 0 \) so small that whenever \( t \leq 1-\delta \) the distance between \( g_j(x, t) \) and \( p \) is at most \( \epsilon \), irrespective of the value of \( x \).) Hence \( H \) is continuous and so \( H \) is a fixed endpoint homotopy that takes \( \delta_0 \) to \( \delta_1 \).

This proof looks complicated until one realizes the geometric meaning of \( G \) and \( H \).
B. Proof That Figure 1 Is Complete

We shall work out the number of different ways in which we can colour the corners of a cube using two colours (black and white say), where two colourings are considered to be the same if 'one is a rotation of the other'. We shall use Polya's Enumeration Theorem. Readers who are unfamiliar with this result are referred to chapter 8 of [Bollobas79].

Each of the 6 faces of a cube has 4 sides, so there are just 24 different rotations which map a cube onto itself. These 24 rotations can be classified as follows:

1. The identity
2. A quarter turn (clockwise or anticlockwise) about an axis passing through the centres of two opposite faces (3×2=6 possibilities)
3. A half turn about an axis passing through the centres of two opposite faces (3 possibilities)
4. A half turn about an axis passing through the mid-points of two diagonally opposite edges (6 possibilities)
5. Rotation through an angle of 2π/3 or -2π/3 about an axis passing through two diametrically opposite corners (4×2=8 possibilities)

The cycle index associated with this group of rotations is \((a_1^6 + 3a_2^2 + 3a_4 + 6a_6 + 8a_8 + 2a_{12})/24\).

Hence if we associate a weight of 1 with all points, black or white, then Polya's Enumeration Theorem shows that the number of different colourings is:

\((2^6+6\times2^2+3\times2^4+6\times2^6+8\times2^8+2^3)/24 = 23\).

Plainly each of the 22 cells in Figure 1 corresponds to a different one of these 23 colourings. The only remaining colouring corresponds to a reflection of cell 11. This proves that any unit cell can be mapped by an appropriate rotation either onto one of the cells in Figure 1 or onto a reflection of cell 11. ■
Chapter 2 - Standard Continuous Analogues and the Euler Characteristic

Introduction

We plan to define the Euler characteristic of a digital picture to be 'the Euler characteristic of its continuous analogue'. But it is not clear what the continuous analogue of a digital picture is - every non-empty normal digital picture has infinitely many continuous analogues. This may seem not to matter very much in the case $d = 2$, for then the Euler characteristic of a boxed polyhedron is equal to the number of its components minus the number of its holes, so that if $C$ is any continuous analogue of a digital picture $(A, S)$ then $\chi(C) = (\text{no. of } \mathcal{A}\text{-components of } S) - (\text{no. of } \mathcal{A}\text{-components of } S^c) + 1$. But in the case $d = 3$ two continuous analogues of a digital picture need not have the same Euler characteristic. Even in the case $d = 2$ the multiplicity of continuous analogues of a digital picture presents a problem in our approach to shrinking (see next chapter). Our objective in this section is therefore to construct a "standard" continuous analogue of the digital picture $(A, S)$, whose definition does not involve too many arbitrary choices. We succeed in doing this for a large class of well-behaved digital pictures.

Elementary Terminology and Special Notations

For all positive integers $N$ let $R_N$ and $Z_N$ respectively denote the set of real numbers modulo $N$ and the set of integers modulo $N$ (thus $R_N = \mathbb{R}/\mathbb{Z}$ and $Z_N = \mathbb{Z}/\mathbb{Z}$, where $x \equiv y$ if and only if $(x-y)$ is an integer multiple of $N$). Note that $Z_N \subseteq R_N$. Let $d$ denote the dimension of the ambient space: thus all our polyhedra are subsets of $\mathbb{R}^d$, and the object point sets of our digital pictures are subsets of $\mathbb{Z}^d$. The value of $d$ is always either 2 or 3. In this chapter and the next $X^c$ denotes $\mathbb{Z}^d \setminus X$ if $X$ is a set of lattice points. Otherwise $X^c$ denotes $R^d \setminus X$. We may assume that terms like 'unit cell', 'window', 'digital picture', and 'A-path' have been suitably re-defined: $R^d$ and $Z^d$ in the original definitions must be replaced by $R^d_N$ and $Z^d_N$. 

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Note that $\mathbb{Z}_n^k$ contains just $N^k$ points, whereas $\mathbb{Z}^k$ contains infinitely many points. So if we work in $\mathbb{R}_n^k$ then for any suitably well-behaved digital picture $(A, S)$ there is no difficulty at all in defining the Euler characteristic both of $(A, S)$ and of $(A, S^c)$. But if we worked in $\mathbb{R}^k$ then one or both of $S$ and $S^c$ would be infinite, and the definitions would require a notion of the "point at infinity", which seems an unnecessary complication. For this reason we have chosen to work in $\mathbb{R}_n^k$ rather than $\mathbb{R}^k$. We usually think of $N$ as being very large, though our results would still hold for fairly small $N$. However, we shall always assume that $N > S$.

If $u$ and $v$ are distinct points in Euclidean space then $uv$ denotes the straight line segment whose endpoints are $u$ and $v$; if $w$ is a point such that $u$, $v$ and $w$ are not collinear then $uvw$ denotes a closed two-dimensional triangle whose corners are $u$, $v$ and $w$. (A straight line segment is defined to include its endpoints.) In $\mathbb{R}_n^k$ there is the problem that there are infinitely many "straight lines" passing through any two distinct points. To avoid unnecessary and irrelevant work, whenever we write $uv$ or $uvw$ it will be assumed that $N$ is so large that the distance between any two of $u$, $v$ and $w$ is less than $(N-1)/3$; then $uv$ denotes the unique shortest line segment with endpoints $u$ and $v$, and $uvw$ denotes the union of $uvuw$ with the smaller of the two regions bounded by $uvuw$ (or just $uvw$ in the degenerate case).

Let $\theta: \mathbb{R}^k \to \mathbb{R}_n^k$ denote the map which takes each $x$ in $\mathbb{R}^k$ to its equivalence class in $\mathbb{R}_n^k$. We say that a subset $P$ of $\mathbb{R}_n^k$ is boxed if $P \in \theta(Q)$ for some closed cube $Q$ whose sides are parallel to the coordinate axes and have length strictly less than $N-1$.

**The Euler Characteristic**

For any non-negative integer $n$, a Euclidean $n$-simplex is the convex hull of $n+1$ points in general position in Euclidean space $\mathbb{R}^k$, and the convex hull of any $r$ of those $n$ points (where $1 \leq r \leq n+1$) is called a face of the simplex. We call a subset $\sigma$ of $\mathbb{R}_n^k$ an $n$-simplex if there is a Euclidean $n$-simplex $\tau$ such that $\theta(\tau) = \sigma$ and $\theta$ is 1-1 on $\tau$.
the image of any face of \( \tau \) is called a face of \( \sigma \).

A complex is a finite collection \( K \) of simplexes such that if \( \sigma \) is in \( K \) then every face of \( \sigma \) is in \( K \), and such that the intersection of any two simplexes \( \sigma \) and \( \tau \) in \( K \) is a common face of \( \sigma \) and \( \tau \). The polyhedron of a complex \( K \), denoted by \( |K| \), is the union of all the simplexes in \( K \). A polyhedron is a set which is the polyhedron of some complex \( K \).

If \( P \) is any polyhedron then the Euler characteristic of \( P \), denoted \( \chi(P) \), is the defined as follows. Let \( K \) be any complex such that \( |K| = P \). Then \( \chi(P) \) is defined to be the alternating sum \( \sum (-1)^n f_n(K) \), where for all non-negative integers \( n \), \( f_n(K) \) denotes the total number of \( n \)-simplexes in \( K \). It can be shown (without too much difficulty) that the value of the alternating sum is independent of the complex \( K \) we choose, so that \( \chi(P) \) is well-defined.

It is plain that, for any two polyhedra \( P \) and \( Q \), \( \chi(P \cup Q) = \chi(P) + \chi(Q) - \chi(P \cap Q) \). It is a deep but well-known result of topology that if \( P \) can be transformed into \( Q \) by "continuous deformation" then \( \chi(P) = \chi(Q) \). This implies that \( \chi(P) = 1 \) if \( P \) can be continuously deformed to a point, in particular, if \( P \) is topologically equivalent to a simplex. One can show quite easily that \( \chi(R^2_N) = 0 \).

**Boxed Polyhedra in \( R^2_N \)**

If \( P \) is a boxed polyhedron in \( R^2_N \) then \( \chi(P) \) is equal to the number of components of \( P \) plus one minus the number of components of \( R^2_N \setminus P \). Thus for boxed polyhedra in \( R^2_N \)

\[
\chi(P) = (\text{no. of components of } P) + (\text{no. of holes for } P) - (\text{no. of components of } R^2_N \setminus P).
\]

**Boxed Polyhedra in \( R^3_N \)**

If \( P \) is a boxed polyhedron in \( R^3_N \) then define \( h_0(P) = \text{number of components of } P \), define \( h_2(P) = (\text{number of components of } R^3_N \setminus P) - 1 \), and define \( h_4(P) = h_0(P) + h_2(P) - \chi(P) \). (Note that \( h_3(P) \) is "the number of cavities in \( P \).") One can prove that \( h_4(P) \) has the
following properties:

(a) \( h_1(P) \) is a non-negative integer

(b) \( h_1(P) = 0 \) if and only if \( P \) is simply connected [KongKosnerKSh]

(c) If \( Q \) is the result of "attaching a solid handle" to \( P \) then \( h_1(Q) = h_1(P) + 1 \)

For these reasons it is natural to think of \( h_1(P) \) as "the number of tunnels in \( P \)." Thus

\[
\text{Euler characteristic} = (\text{no. of components}) + (\text{no. of cavities}) - (\text{no. of tunnels}).
\]

CLOSED DIGITAL CURVES

If \( A \) is an adjacency relation, then we define a closed \( A \)-curve to be a finite set \( R \) of lattice points equipped with a symmetric relation \( \ast \) such that: (a) \( u \ast v \) implies \( u \) is

\( A \)-adjacent to \( v \); (b) for all \( r \) in \( R \) there are just two points \( x \) in \( R \) such that \( r \ast x \); and

(c) \( R \) is \( \ast \)-connected in the sense that if \( R = R_1 \cup R_2 \) is any partition of \( R \) then there exists \( r_1 \in R_1 \) and \( r_2 \in R_2 \) such that \( r_1 \ast r_2 \). We call \( R \) the point-set of the closed

\( A \)-curve \((R,\ast)\), and we call \( \ast \) the adjacency relation of \((R,\ast)\). The length of a closed

\( A \)-curve \((R,\ast)\) is the number of elements in \( R \). We define a simple closed \( A \)-curve to be

a closed \( A \)-curve \((R,\ast)\) such that \( u \ast v \) if and only if \( u \) is \( A \)-adjacent to \( v \).

For brevity we will not usually distinguish between a simple closed \( A \)-curve and its

point-set.

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**Standard Analogues of Digital Pictures**

We are now ready to define standard continuous analogues.

---

**A. The Two-Dimensional Case**

Let \( C_0 = S \); let \( C_1 \) be the union of all straight line segments whose endpoints are \( A \)-adjacent points in \( S \); let \( C_2 \) be the union of all \( 1 \times 1 \) squares and \((1,1,\sqrt{2})\) triangles whose sides are all contained in \( C_1 \). Define \( C^0(A,S;Z) = C_0 \cup C_1 \cup C_2 \). For brevity we will
usually abbreviate $C^*(A,S;2)$ to $C^*(A,S)$ when $d = 2$.

B. The Three-Dimensional Case

NOTATION

Let $G$ be a simple closed $A$-curve contained in a unit cell $K$; then $(K,G)$ is of type 6, 7, 8, 9, 11, 12, 13, 14, 15, or 18. If $G$ is not contained in a face of a unit cell (i.e. if $(K,G)$ is not of type 6 or 9) then we define $\sigma (A,G)$ to be the set of all the $(1, (\sqrt{3})/2, (\sqrt{3})/2)$ and $(\sqrt{2}, (\sqrt{3})/2, (\sqrt{3})/2)$ triangles with one corner at the centroid of $K$ whose two other corners are $A$-adjacent points in $G$. 

Let $C^*(A,S;3)$ denote the smallest polyhedron in $R^3$ such that:

(a) $S \subseteq C^*(A,S;3)$

(b) Each straight line segment that joins two $A$-adjacent points in $S$ is contained in $C^*(A,S;3)$

(c) If $K$ is any unit cell and $G$ is a simple closed $A$-curve in $S\hat{K}$ such that $G$ is not contained in any face of $K$, and $T \in \sigma (A,G)$, then $T \subseteq C^*(A,S;3)$ if there do not exist two $A$-adjacent points in $K\hat{S}^C$ such that the straight line segment joining those corners meets $T$

(d) Let $C_2(A,S)$ denote the smallest polyhedron in $R^3$ that satisfies (a), (b) and (c). Then for all unit cells $K$, and all faces $F$ of $K$, each component of $K\setminus C_2(A,S)$ that does not contain a corner of $K$ is a subset of $S$, and each component of $F\setminus C_2(A,S)$ that does not contain a corner of $K$ is also a subset of $C^*(A,S;3)$

For brevity we will usually write $C^*(A,S)$ as an abbreviation for $C^*(A,S;3)$ when $d = 3$.

Notice that $C^*(\ldots ;d)$ is 'monotonic', in the sense that if $S \subseteq T \subseteq Z^d$ and $A$ and $A'$ are
adjacency relations on $\mathbb{Z}_n^d$ such that $A = A'$ on $S$ then $C^*(A,S) \leq C^*(A,T)$. Our next task is to establish simple general conditions under which $C^*(A,S)$ is a continuous analogue of $(A,S)$. Notice also that condition (c) would still hold if we replaced the second "if" by "if and only if".

REMARKS
If $(A,S)$ is strongly normal then so are $(A,S^X)$, $(A,S^G)$ and $(A,W)$, where $X$ is any window and $W$ is any union of $A$-components of $S$. (c.f. Proposition 1(ii) in Chapter 2).

We also observe that in the case $d = 2$, a $(A,S)$ has no continuous analogue at all if $(A,S)$ is not strongly normal. ☐

DEFINITION
Let $X$ be a polyhedron in $\mathbb{R}^n$. We define a regular dissection of $X$ to be a complex $M$ in which every 0-simplex is (a singleton set containing) either a lattice point or the centroid of a unit cell, such that $|M| = X$ and every simplex of $M$ is contained in some unit cell. We define a strongly regular dissection of $X$ to be a regular dissection of $X$ in which every 0-simplex is a (singleton set containing) a lattice point. (Not every polyhedron in $\mathbb{R}^n$ has a regular dissection.) ☐

PROPOSITION 1
(i) Let $X$ be any window. Then $C^*(A,S^X) = C^*(A,S)^X$, and $C^*(A,S)^X$ is a continuous analogue of $(A,S)$ relative to $X$ if and only if, for all unit cells $K$ in $X$,

$C^*(A,S)^K$ is a continuous analogue of $(A,S)$ relative to $K$

(ii) If $(A,S)$ is strongly normal then $C^*(A,S)$ is a continuous analogue of $(A,S)$

PROOF
It is plain that if $X$ is any window then $C^*(A,S^X) = C^*(A,S)^X$. It is immediate from the definition of a continuous analogue relative to a window that if, for all unit cells $K$ in a window $X$, the polyhedron $C^*(A,S)^K$ is a continuous analogue of $(A,S)$ relative to $K$ then
C*(A,S)NK is a continuous analogue of (A,S) relative to X. Moreover, if X is any window and C = C*(A,S)NK satisfies conditions (iii) and (v) in the definition of a continuous analogue of (A,S) relative to X then, for all unit cells K in X, C = C*(A,S)NK must satisfy the corresponding conditions in the definition of a continuous analogue of (A,S) relative to K.

In complete the proof of (i), we assert that, for all unit cells K and all digital pictures (A,S), C = C*(A,S)NK satisfies conditions (i), (ii) and (iv) in the definition of a continuous analogue of (A,S) relative to K. In fact the validity of conditions (i) and (iv) is obvious, so we have only to confirm the validity of condition (ii). Plainly each component of C*(A,S)NK contains a corner of K, so if the assertion it remains to show that given any two corners x, y of a unit cell K, x and y belong to the same component of C*(A,S)NK if and only if x and y belong to the same A-component of SNK. It is plain that 'if' is true. The 'only if' part is obviously true (by the definition of C*) if SNK does not contain a simple closed A-curve Γ that is not contained in a face of K. If SNK does contain a simple closed A-curve Γ that is not contained in a face of K then the 'only if' part still holds, for there is at most one point in SNK that is not 6-adjacent to Γ, so if z is a corner of K that is not in the same A-component of SNK as Γ then (z) is both an A-component of SNK and a component of C*(A,S)NK.

We have now established part (i) of the proposition, and we also know that C = C*(A,S) satisfies conditions (i), (ii) and (iv) in the definition of a continuous analogue of (A,S). From now on we assume that (A,S) is a strongly normal digital picture, and K is an arbitrary unit cell.

We claim that C = C*(A,S)NK satisfies condition (v) in the definition of a continuous analogue of (A,S) relative to K: we claim that a component P of C*(A,S)NK meets the frontier of a component Q of C*(A,S)CNK if and only if PAQ* is A-adjacent to QAQ*. The fact that (A,S) is (strongly) normal implies that either SNK is A-connected (whence
$C^*(A, S)^{\gamma K}$ is connected; or $S^{\gamma K}$ is $\mathcal{A}$-connected (whence $C^*(A, S)^{\gamma K}$ is connected); or (for $d = 3$ only) $(S, K)$ is of type 14 and each of $S\mathcal{K}$ and $S^{\gamma K}$ has just two $A$-components, one consisting of just one point and the other consisting of three pairwise $A$-adjacent points.

(This is plain when $d = 2$, and when $d = 3$ it follows from Proposition 1(iii) in Chapter 2.)

In the first case the unique component of $C^*(A, S)^{\gamma K}$ must meet the frontier of every component of $C^*(A, S)^{\gamma K}$, while the unique $A$-component of $S\mathcal{K}$ must be $A$-adjacent to every $A$-component of $S^{\gamma K}$. So our claim holds in the first case. Similarly our claim holds in the second case, and when $d = 3$ we see by inspection that the claim holds in the third case.

It remains to show that $C = C^*(A, S)^{\gamma K}$ satisfies condition (iii) in the definition of a continuous analogue of $(A, S)$ relative to $K$. First of all, we observe that given any two corners $x, y$ of a unit cell $K$, $x$ and $y$ certainly belong to the same component of $C^*(A, S)^{\gamma K}$ if they belong to the same $A$-component of $S^{\gamma K}$. (This depends on the fact that $(A, S)$ is strongly normal in the case $d = 2$, but not in the case $d = 3$.) Since it is clear that each component of $C^*(A, S)^{\gamma K}$ contains a corner of $K$, it remains only to establish that if two corners of $K$ belong to the same component of $C^*(A, S)^{\gamma K}$ then they belong to the same $A$-component of $S^{\gamma K}$. So suppose two corners $x$ and $y$ of $K$ belong to the same component of $C^*(A, S)^{\gamma K}$. It is plain (by inspection) that $x$ and $y$ must belong to the same $A$-component of $S^{\gamma K}$ if $d = 2$. This means that the Proposition holds in the case $d = 2$.

Now suppose $d = 3$. If there are two diametrically opposite corners of $K$ that are in $S^{\mathcal{R}}$ and are $A$-adjacent to each other then $S^{\gamma K}$ is $A$-connected, so $x$ and $y$ must belong to the same $A$-component of $S^{\gamma K}$, as required. Hence we may assume that such a pair of corners does not exist. Let $A'$ be the adjacency relation such that $A' = A$ except that two points which are not $A'$-adjacent are not $A'$-adjacent (even if they are $A$-adjacent), and if $F$ is any face of $K$ that contains just two diagonally opposite points in $S$, and the two corners of $F$ that are in $S^{\mathcal{R}}$ are $A$-adjacent, then the corners of $F$ that are in $S$ are not
A'-adjacent (even if they are A-adjacent). Suppose, for the purpose of deriving a contradiction, that \(x \) and \(y \) belong to different A-components of \(S^nK \).

Then \(x \) and \(y \) belong to different A'-components of \(S^nK \). It is plain that, when regarded as a two-dimensional digital picture (in the generalized window \(Bd K \)), \((A', S^nBd K) \) is strongly normal. Moreover, it is readily confirmed that \((Bd K) \cap C(A', S) = C(A', S^nBd K; Z) \). So since the Proposition holds in the case \(d = 2 \), and since \(x \) and \(y \) belong to different A'-components of \(S^nBd K \), we know that \(x \) and \(y \) must belong to different components of \((Bd K) \cap C(A', S)\). Let \(L \) be a strongly regular dissection of \(C(A', S) \) and \(S^nBd K \). Since \(|L| \) separates \(x \) from \(y \) on \(Bd K \), it is easy to deduce from the Jordan Curve Theorem that there must exist a simple closed curve \(\gamma \) on \(Bd K \) such that the point-set \(C \) of \(\gamma \) is a union of 1-simplices of \(L \) and \(x \) and \(y \) belong to different components of \((Bd K) \setminus C \). We assume WLOG that \(\gamma \) is a shortest possible curve with this property.

Let \(s \) and \(t \) be any two points in \(S^nK \) such that the straight line segment \(st \) is contained in \(C \). Then \(s \) and \(t \) are A'-adjacent. Let \(\leftrightarrow \) be the relation on \(G_nZ^3 \) defined by \(u \leftrightarrow v \) iff the straight line segment \(uv \) is contained in \(C \). Then \((G_nZ^3, \leftrightarrow) \) is a closed A'-curve, and \(s \leftrightarrow t \). Hence there exists a closed A'-curve \((R, \sim) \) of minimal length such that \((a) \ s \sim t \) and \((b) \) there is at most one pair of points \(\langle p, q \rangle \) such that \(p \sim q \) and \(\neg (p \leftrightarrow q) \). Then since \((R, \sim) \) is of minimal length it must be a simple closed A'-curve. Moreover \(K \) cannot be contained in a face of \(K_\sim \) for if it were then -- bearing in mind that there is at most one pair of points \(\langle p, q \rangle \) such that \(p \sim q \) and \(\neg (pq \in C) \) -- we see that \(\gamma \) would not be as short as possible (check).

Since \(s \) is A'-adjacent to \(t \), it follows from the definition of A' that there is no straight line segment joining two 18-neighbours \(w \) and \(z \) in \(S^e \) such that \(wz \) meets \(st \). By our earlier assumption there do not exist two diametrically opposite corners of \(K \) that are in \(S^e \) and are A-adjacent to each other. So it follows from the definition of \(C^e(A, S) \) that the triangle \(\Delta tvg \) is contained in \(C^e(A, S) \), where \(g \) is the centroid of \(K \).
But s and t were arbitrary corners of K such that st ∈ G. So since x and y are in different components of (Bd K)\G it follows from the (intuitively obvious) Proposition 0 in Chapter 5 that x and y are in different components of K\C*(A,S). ■

COROLLARY

If (α,β) ≠ (β,β) then C*(α,β,S) is a continuous analogue of (α,β,S). Also, C*(A₁',S) is a continuous analogue of (A₁',S). (Recall that A₁' is the first of the two adjacency relations introduced in the last chapter to represent the face centred cubic lattice.)

PROOF

Let D be any unit cell or any face of a unit cell. Then unless (α,β) = (β,β) either SND is α-connected or SND is β-connected. So (α,β,S) is strongly normal if (α,β) ≠ (β,β). It is plain that A₁' is strongly normal. ■

DEFINITION

If C*(A,S) is a continuous analogue of (A,S) then we call it the standard analogue of (A,S). ■

**Consistent Families of Digital Pictures**

The theory of digital pictures introduced in Chapter 2 is a static theory, in the sense that the picture (A,S) is regarded as being fixed. In order to discuss shrinking and thinning, we must allow S to change. Furthermore, A must be allowed to change with S, since this is what happens during shrinking when we use 2E-adjacency for the object points and 6-adjacency for the background points, say.

For brevity we shall often use (A( ), ) to denote a family ((A(S),S)|S∈2₆−₂₃₆| of digital pictures.
DEFINITION
The family \((A(.))\) is consistent if, for all \(p \in S \subseteq \mathbb{Z}_n^d\), \(A(S \setminus \{p\}) = A(S)\) on \(\mathbb{Z}_n^d \setminus \{p\}\).  

DEFINITION
A family \((A(.))\) is strongly normal if \((A(S), S)\) is strongly normal for all \(S \subseteq \mathbb{Z}_n^d\).  

NOTATION
Suppose \(d = 3\). If each of \(\alpha\), \(\beta\) and \(\delta\) is equal to 6, 18 or 26 then let \((\alpha, \beta, \delta, \ldots)\) denote the family of all digital pictures in which two object points are adjacent iff they are \(\alpha\)-adjacent, two background points are adjacent iff they are \(\beta\)-adjacent, and an object point is adjacent to a background point iff the two points are \(\delta\)-adjacent. Let \((\alpha, \beta, \ldots) = (\alpha, \beta, \delta, \ldots)\), and let \((\alpha, \ldots) = (\alpha, \alpha, \alpha, \ldots)\).  

REMARKS
It is plain that \((\alpha, \beta, \delta, \ldots)\) is a consistent family. Also, \((\alpha, \beta, \delta, \ldots)\) is strongly normal unless \(\alpha = \beta = \delta\). Observe that if \((A(.))\) is a consistent family and \(S \subseteq T \subseteq \mathbb{Z}_n^d\) then \(A(S) = A(T)\) on \(S \cup \mathbb{Z}_n^d\), and \(C^*(A(S), S) \subseteq C^*(A(T), T)\). If \(\{C(A(S), S) | S \subseteq \mathbb{Z}_n^d\}\) is consistent then so are the families \(\{C(A(S)^c, S) | S \subseteq \mathbb{Z}_n^d\} = \{C(A(S), S^c) | S \subseteq \mathbb{Z}_n^d\}\) and \(\{C(A(S \cup W), S) | S \subseteq \mathbb{Z}_n^d\}\) where \(W\) is any given set.  

Euler Characteristic

DEFINITION
The Euler Characteristic of a strongly normal digital picture \((A, S)\) is defined to be the Euler characteristic of \(C^*(A, S)\), and is denoted by \(X(A, S)\).
REMARK

When $d = 2$, if $(A, T)$ is any strongly normal boxed digital picture then $X(A, T)$ is just one greater than the number of $A$-components of $T$ minus the number of $A$-components of $T^c$. □

A USEFUL IDENTITY

Let $C$ be any polyhedron in $R^d_n$, and let $K$ be any unit cell. Let $K^0$ denote the set of corners of $K$, let $K^1$ denote the union of the edges of $K$ and, in the case $d = 3$, let $K^2$ denote the union of the faces of $K$. We define:

$$X(C; K) = X(CN K) - X(CN K^1)/2 - X(CN K^0)/4$$

if $d = 2$

and

$$X(C; K) = X(CN K) - X(CN K^2)/2 - X(CN K^1)/4 - X(CN K^0)/8$$

if $d = 3$.

Let $L$ be a complex such that $|L| = C$, and such that every simplex in $L$ is contained in some unit cell. By applying a straightforward inclusion-exclusion argument to the standard formula expressing $X(C)$ in terms of the simplexes of $L$, we deduce the following identity:

If $n$ is any positive integer and $C$ is any polyhedron in $R^d_n$

then $X(C) = \sum \{X(C; K) \mid K$ is a unit cell in $R^d_n\}$. □

This identity is the principal reason for our interest in the number $X(C; K)$. Besides being useful in mathematical arguments it is good for computing the Euler characteristic of digital pictures, as will be explained below.

REMARKS

1. Polyhedra in Windows

Let $X$ be any window in $R^d_n$. It is plain that $X(X; K) = 0$ for all unit cells $K$ contained in $X$. Hence $X(X) = \sum X(X; K) \mid K$ is not contained in $X$. So if $C$ is any polyhedron such that $X \subset C \not\subset X$ then, since $X(C; K) = X(X; K)$ for all unit cells $K$ that are not contained in $X$, $X(C) = \sum X(C; K) \mid K$ is a unit cell in $R^d_n$. □

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\( X(C) + \sum \chi(C; K) \mid K \in X \) . (**) This generalization of (*) is sometimes useful for computing the Euler characteristic of the complement of a digital picture in a given window.

2. Polyhedra in Euclidean Space (with point at infinity).

Suppose \( C \) is a polyhedron in \( \mathbb{R}^d \), where \( \infty \) is the "point at infinity". (In this context a polyhedron is either a bounded polyhedron in \( \mathbb{R}^d \), or the union of the singleton set \( \{\infty\} \) with any finite union of finite intersections of closed half-spaces of \( \mathbb{R}^d \).) Note that, unlike every other point, the point \( \infty \) is not contained in any unit cell. If \( C \) is bounded (i.e., if \( C \) does not contain the point \( \infty \) then it is plain that the identity (**) is still true when \( \mathbb{R}^d \) is replaced by \( \mathbb{R} \), since \( C \) meets only finitely many unit cells and for all other unit cells \( K \) we have \( \chi(C; K) = 0 \). Thus \( X(C) = \sum \chi(C; K) \mid K \in \mathbb{R}^d \) if \( C \) is bounded.

Another important special case is that in which \( \mathbb{R}^d \) is bounded. Let \( X \) be a square (\( d = 2 \)) or cubical (\( d = 3 \)) window that contains \( \mathbb{R}^d \). Then \( X(C) = X(C \cap X) + X(C \setminus \text{int}(X)) \), \( X(Fr X) = X(X) + \sum \chi(C; K) \mid K \subseteq X \) + \( X(C \setminus \text{int}(X)) \) \( - X(Fr X) \) (by (**) ). But \( X \) and \( C \setminus \text{int}(X) \) are polyhedral d-balls, so \( X(X) = X(C \setminus \text{int}(X)) = 1 \). Hence \( X(C) = \sum \chi(C; K) \mid K \subseteq X \) + 2 \( - X(Fr X) \). Now \( \sum \chi(C; K) \mid K \subseteq X \) = \( \sum \chi(C; K) \mid K \subseteq \mathbb{R}^d \), since \( K \subseteq C \) for all \( K \) not contained in \( X \), whence \( X(C; K) = 0 \) for all \( K \) not contained in \( X \). Moreover, \( X(Fr X) = 0 \) or 2 according as \( d = 2 \) or 3. This shows that if \( C \) is unbounded then \( X(C) = 2 + \sum \chi(C; K) \mid K \subseteq \mathbb{R}^d \) if \( d = 2 \), and \( X(C) = \sum \chi(C; K) \mid K \subseteq \mathbb{R}^d \) if \( d = 3 \).  

Table 1 lists the values of \( X(C^*(A, S); K) \) by the type of \( (S, K) \) for all strongly normal two-dimensional digital pictures, and also tabulates \( X(C^*(6, 18, S); K) \) and \( X(C^*(18, 6, S); K) \).

There is very little point in tabulating \( X(C^*(6, 26, S); K) \) and \( X(C^*(26, 6, S); K) \) in addition to \( X(C^*(6, 18, S); K) \) and \( X(C^*(18, 6, S); K) \), since \( X(C^*(6, 26, S); K) = X(C^*(6, 18, S); K) \) except when \( (K, S) \) is of type 18 (in which case \( X(C^*(6, 26, S); K) = -3/4 \)), and \( X(C^*(26, 6, S); K) = X(C^*(18, 6, S); K) \) except when \( (K, S) \) is of type 5 (in which case \( X(C^*(26, 6, S); K) = -3/4 \)).
It follows from Table 1 and the remarks in the last paragraph that, in the case $d = 2$,

$$X(C^*(A,S);K) = \overline{X}(C^*(A,S^c);K)$$

for all strongly normal $(A,S)$; in the case $d = 3$,

$$X(C^*(G,18,S);K) = X(C^*(G,26,S^c);K)$$

and $X(C^*(G,26,S);K) = X(C^*(26,6,S^c);K)$. Hence when $d = 2$ $X(A,S) = \overline{X}(A,S^c)$ for all strongly normal $(A,S)$, and when $d = 3$

$$X(G,18,S) = X(18,6,S^c)$$

and $X(G,26,S) = X(26,6,S^c)$.

An Example - The Jordan Curve Theorem for Digital Curves

Let $S$ be a simple closed $A$-curve in the plane such that $(A,S)$ is strongly normal. Let $G = C^*(A,S)$, and let $H = C^*(A,S^c)$ (in this example $S^c$ denotes $Z^\infty \setminus S$). Then by the above remark $X(H) = 2 + \sum (X(H;K) \mid K \text{ is a unit cell})$ ($\ddagger$). But by the observation in the last paragraph $X(L^*(A,S);K) = \overline{X}(C^*(A,S^c);K)$. In other words $X(G;K) = \overline{X}(H;K)$. Hence ($\ddagger$) implies that $X(H) = 2 - X(G)$. Since $G$ is a simple closed polygonal curve $X(G) = 0$. So

$$X(H) = 2.$$ Therefore $H$ is not connected, since a connected polyhedron has Euler characteristic at most 1. It follows that $S^c$ is not $A$-connected. It is readily confirmed that, for all $p$ in $S$, $S^c N(p)$ has just two $A$-components. So $S^c$ cannot have more than two $A$-components because each $A$-component of $S^c$ must be $A$-adjacent to $S$. ■

| Table 1: Values of $X(C^*(A,S);K)$ for Different Types of $(S,K)$ |
|-------------|------|-------|      |      |      |      |
| $d = 2$     |      |      |      |      |      |      |
| Type of $(K,S)$ | 0   | 0    | 1    | 1    | 1    | 0    | 1    | 1    |
|              | 0   | 0    | 0    | 0    | 0    | 1    | 0    | 1    |
| Value of $X(C^*(A,S);K)$ | 0 | 1/4 | 0 | 1/2 or $-1/2$* | $-1/4$ | 0 |

* This assumes that $(A,S)$ is strongly normal, in which case if $X(C^*(A,S);K) = 1/2$ then $X(C^*(A,S^c);K) = -1/2$, and vice versa.
### $d = 3$

<table>
<thead>
<tr>
<th>Type of $(k,s)$</th>
<th>Value of $X(C^*(6,18,5); K)$</th>
<th>Value of $X(C^*(18,6,5); K)$</th>
</tr>
</thead>
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<tr>
<td>1</td>
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<td>0</td>
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</tr>
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</tr>
<tr>
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<td>$-1/4$</td>
</tr>
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<td>$-1/8$</td>
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<tr>
<td>22</td>
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</tbody>
</table>
Computing Euler Characteristics of Digital Pictures

Several different approaches to computing Euler characteristics of digital pictures have been suggested in the literature. Gray [Gray71] proposed a method for two-dimensional pictures based on the identity \( X(P) = (\text{no. of west-facing convexities of } P \text{ minus the number of west-facing concavities of } P) \). (Examples: A circular disc has exactly one west-facing convexity and no concavities. An annulus has one west-facing convexity and one west-facing concavity, the latter being on the "inside" of the hole.) In [BieriNef84] two recursive algorithms were given for computing the Euler characteristics of any \( P \) which is a union of \( d \)-dimensional unit cells, based respectively upon the identities: 

\[
\begin{align*}
\sum f_k(P) &= \sum_{\sigma} (\text{cut}(\sigma H^k) - \text{cut}(\sigma H^{k-1})) \\
X(P) &= \sum_{\sigma} (\text{cut}(\sigma H^0) - \text{cut}(\sigma H^{-1})).
\end{align*}
\]

Here \( \sigma \) is the plane \( x = m \) and \( H^0 \) is the plane \( x = m + 1/2 \). In the first identity \( f_k(P) \) is the number of \( k \)-dimensional unit cells in \( P \), so that depending upon whether \( d = 2 \) or \( d = 3 \), \( X(P) = (f_0(P) - f_1(P) + f_2(P)) \) or \( (f_0(P) - f_1(P) + f_2(P) - f_3(P)) \).

While these approaches are interesting, it is not clear that they are in general better than the more obvious approach of adding up the contribution of each unit cell to the the alternating sum used to define \( X \). Such a method was also given in [Gray71]. In [YokoiToriwakiFukumura73], the authors gave a related method: a "coefficient of curvature" was defined for each object point. It was proved in [YokoiToriwakiFukumura73] (Theorem 9) that at each border point \( p \), \( 2\pi \) times the coefficient of curvature is equal to the signed angle change (positive = anti-clockwise) experienced by a "left-hand on wall" border following algorithm as it passes through \( p \) (we add up all the angle changes if the algorithm passes through \( p \) more than once). It can be seen from this that the Euler characteristic of the set of object points is equal to the sum of the coefficients of curvature at the border points. (We observe that this result can be generalized to three dimensions by appealing to a well-known formula that is sometimes attributed to Descartes: given a polyhedronal surface \( S \) in Euclidean space define the coefficient of curvature at each vertex \( v \) of \( S \) to be \( 2\pi \) minus the sum of the face angles incident upon \( v \) -- then the sum
of the coefficients of curvature at all the vertices of $S$ is equal to $2\pi$ times the Euler characteristic of $S$.)

Lobregt et al. [LobregtVerheestVerbeeck2011] used the "add up the contributions of each unit cell" approach to compute the Euler characteristic of the surface of a union of three-dimensional unit cells. Lobregt et al. used the symbol $N_{e}^{(c)}$ to denote the contribution of a unit cell to the surface characteristic. Here $c$ is a numerical 'encoding' of the (object,background) configuration in a unit cell. To calculate $c$, first number the eight voxels in the cell from 0 to 7 as follows:

\[
\begin{array}{ccc}
0 & 1 & 4 \ 5 \\
2 & 3 & 6 \ 7 \\
\end{array}
\]

Plane 1 \hspace{1cm} Plane 2

Then $c = 2^{0} \langle 2^{n} \mid \text{voxel } n \text{ is an object voxel} \rangle$. (For example, $c = 0$ encodes a cell of type 1, while $c = 2^{0}$ (for any $n$ in the range $0 \leq n \leq 7$) encodes a cell of type 2, and $c = 255$ encodes a cell of type 22.) Plainly $0 \leq c \leq 255$. Care is needed with cells of type 19 (the cases $c = 126, 185, 219$ or 231). To arrive at the correct value of $N_{e}^{(c)}$ one needs the concept of a 'degenerate tunnel' -- see [TsaoFu81] p319 Example 4.

It turns out that $X(C^{*}(G,26,5);K) \equiv N_{e}^{(c)}/2$, where $c$ is the code of the configuration $(S^{3}K,S^{3}K)$. This identity is related to the theorem that the Euler characteristic of a 3-manifold with boundary is one half the Euler characteristic of its boundary.

Lobregt et al. also tabulate the quantity $N_{26}^{(c)}$, where $N_{26}^{(c)}$ is by definition equal to $N_{e}^{(c)}$. Now $(255 - c)$ is of course the 'ones-complement' of $c$, so the configuration which $(255 - c)$ encodes is obtained from the configuration $c$ encodes by swapping object and background voxels.) This definition of $N_{26}^{(c)}$ should be contrasted with the fact that in our theory $\langle X(C^{*}(G,26,5);K) = X(C^{*}(26,G,5);K) \rangle$ is not a definition but is a theorem.
We propose that the identity \( X(C) = \{ X(C;K) | K \text{ is a unit cell in } \mathbb{R}^n \} \) be used to compute \( X(A,S) \). In the case of \((6,2S,)\) and \((2S,6,6,)\) it is plain from the above remarks that this is essentially the same as the algorithm of Lobregt et al. As in their algorithm, the values of \( X(C^*(A,S);K) \) for each of the 256 \((S_{N}K,S_{N}K)\)-configurations can be stored in a look-up table to make the algorithm faster. The simplest method of using the identity is to perform a raster scan, during which every unit cell in \( \mathbb{R}^n \) is considered. But \( X(C^*(A,S);K) = 0 \) if every corner of \( K \) is in \( S \) or if every corner of \( K \) is in \( S^c \); so when computing \( X(A,S) \) by means of the identity we only need to look at unit cells that contain two \( S \)-adjacent corners of which one is in \( S \) and the other is in \( S^c \). All such cells can be found by a border-tracking algorithm, and for sufficiently large digital pictures with few object and background components, and few holes this approach will tend to be faster than raster scanning.

**The Labelled Adjacency Graph and \( X \)-Regularity**

The adjacency graph tells us nothing about the Euler characteristic of a three-dimensional digital picture. It is therefore natural to label each \( S \)-vertex \( u \) of \( \text{adj}(A(S),X,S) \) with 'the Euler characteristic of COMPT(\( u \))', and to give some similar label to each \( S^c \)-vertex. However, it is not quite clear whether we want 'the Euler characteristic of COMPT(\( u \))' to mean '\( X(A(S),\text{COMPT}(u)) \)', or '\( X(P) \) where \( P \) is the component of \( C^*(A(S),SNX) \) such that \( P \cap \mathbb{Z}^n = \text{COMPT}(u) \)', or '\( X(A(\text{COMPT}(u)),\text{COMPT}(u)) \)'. The last choice may seem a little less natural than the first two, but on the other hand it does have the advantage of depending only upon COMPT(\( u \)) and \( A(\cdot) \), whereas the other choices might also be influenced by \( S^c \setminus \text{COMPT}(u) \).

There is a similar dilemma in our choice of labels for the \( S^c \)-vertices of \( \text{adj}(A(S),X,S) \).

It is convenient to give a name to the class of picture families for which no problem arises:
DEFINITION

We say \((A(\cdot, \cdot))\) is \(X\)-regular if \((A(\cdot, \cdot))\) is strongly normal and, for all \(S \in \mathbb{Z}_n^d:\)

\[
X(A(PnZ_n^d), PnZ_n^d) = X(A(S), PnZ_n^d) = X(P) \quad \text{for all components } P \text{ of } C^*(A(S), S) \\
X(A(Q'nZ_n^d), Q'nZ_n^d) = X(A(S), Q'nZ_n^d) = X(Q') \quad \text{for all components } Q' \text{ of } C^*(A(S), S^c) \\
X(A(P'nZ_n^d), P'nZ_n^d) = \pm X(P') \quad \text{for all components } P' \text{ of } C^*(A(S), S^c) \\
X(A(Q'nZ_n^d), Q'nZ_n^d) = \pm X(Q) \quad \text{for all components } Q \text{ of } C^*(A(S), S^c)
\]

where the negative signs obtain if \(d = 2\), and the positive signs if \(d = 3\).

* Observe that since \(C^*(A(S), S)\) is a continuous analogue of \((A(S), S)\), \(PnZ_n^d\) and \(P'nZ_n^d\)
are \(A(S)\)-components of \(S\), while \(Q'nZ_n^d\) and \(Q'nZ_n^d\) are \(A(S)\)-components of \(S^c\).

Note that the conditions "\(X(A(PnZ_n^d), PnZ_n^d) = \pm X(P)\)" and "\(X(A(Q'nZ_n^d), Q'nZ_n^d) = \pm X(Q)\)"
are discrete forms of 'Alexander Duality'.

We shall show in Proposition 3 that the picture families we are most interested in (from
the viewpoint of practical applications) are all \(X\)-regular.

PROPOSITION 2

Suppose \((A(\cdot, \cdot))\) is \(X\)-regular. Then:

(i) The family \(\{X(A(S), S^c) \mid S \in \mathbb{Z}_n^d\}\) is \(X\)-regular

(ii) For all \(S \in \mathbb{Z}_n^d\), \(X(A(S), S) = \Sigma X(W, W) \mid W\) is an \(A(S)\)-component of \(S\).\)

(iii) If \(d = 3\) then for all \(S \in \mathbb{Z}_n^d\), \(X(A(S), S) = X(A(S), S^c)\). If \(d = 2\) then, for all
\(S \in \mathbb{Z}_n^d\), \(X(A(S), S) = -X(A(S), S^c)\).

PROOF

(i) is obvious. (ii) follows from the following chain of equalities:

\[
X(A(S), S) = X(C^*(A(S), S)) = \Sigma X(P) \mid P \text{ is a component of } C^*(A(S), S)\)
\]
\[ \Sigma \{ X(A(Pn^2), Pn^2) \mid P \text{ is a component of } C'(A(S), S) \} \text{ by } \chi \text{-regularity} \]
\[ = \Sigma \{ X(A(W), W) \mid W \text{ is an } A(S)\text{-component of } S \}. \]

Now if \( B \) and \( U \) are disjoint subsets of \( R_n^* \) such that \( B^c \) and \( D^c \) are polyhedra then
\[ X(B^c) + X(D^c) = X(B^c \cap D^c) + X(B^c) + X(D^c) = X(B^c) + X(D^c). \]
By induction on \( n \) we deduce that
\[ x((Q_1, Q_2, \ldots, Q_d)^c) = \Sigma x(Q_i^c) \text{ if the } Q_i \text{ are disjoint} \]
and each \( Q_i^c \) is a polyhedron.

Hence (iii) follows from the chain of equalities
\[ X(A(S), S^c) = \Sigma \{ X(A(W), W) \mid W \text{ is an } A(S)\text{-component of } S^c \} \text{ by (ii)} \]
\[ = \Sigma \{ X(Q^c) \mid Q \text{ is a component of } C'(A(S), S)^c \} \text{ by } \chi \text{-regularity} \]
\[ = \Sigma X(C'$A(S), S)) \text{ by the above argument} \]
\[ = \pm X(A(S), S), \]
where the positive sign obtains for \( d = 3 \) and the negative sign for \( d = 2 \). \[ \blacksquare \]

**DEFINITION**

Let \((A(.), .)\) be a \( \chi \)-regular family of digital pictures, and let \( S \) be any subset of \( R_n^* \).

The labelled adjacency graph of \((A(.), S)\) relative to a window \( X \), denoted by
\( ADJ(A(.), X, S) \), is defined to be the labelled graph obtained when each \( S \)-vertex \( u \) of
\( ADJ(A(.), X, S) \) is labelled with the value of \( X(A(S), COMPT(u)) \), and each \( S^c \)-vertex \( u \) of
\( ADJ(A(.), X, S) \) is labelled with the value of \( X(A(S), COMPT(v)) \). The labelled
adjacency graph of \((A(.), S)\), denoted \( ADJ(A(.), S) \), is the labelled adjacency graph of
\((A(.), S)\) relative to \( R_n^* \). \[ \blacksquare \]

**REMARK**

We defined \( ADJ(A(.), S) \) to be a vertex-labelled graph. It would also have been possible to encapsulate the information conveyed by \( ADJ(A(.), S) \) in an edge labelled graph. If \( ADJ(A(S), S) \) is a tree then the edge labelled representation will be marginally
more compact, since the number of edges of a tree is exactly one less than the number of vertices.

*To construct an ‘edge-labelled version of $\text{ADJ}(A(.),S)$’, we observe that given any vertex-labelled bipartite graph $G$ such that the sum of the labels on each vertex class is the same, one can label the edges of $G$ in such a way that the label on each vertex is equal to the sum of the labels on the edges incident upon that vertex. In the case when $G$ is a tree, this result is easily proved by induction on the number of edges of $G$.

In the general case, let $T$ be a spanning tree of $G$; assign a label of 0 to each edge of $G$ that is not an edge of $T$, and then apply the result for $G = T$.

Now the sums of the labels on the two vertex classes of $\text{ADJ}(A(.),S)$ (the $S^-$ and $S^+$-vertices) are the same if $d = 3$, and are the same in magnitude but opposite in sign if $d = 2$ (since $(A(.),.)$ is $X$-regular). Hence it is possible to label the edges of $\text{ADJ}(A(.),S)$ in such a way that the sum of the labels on the edges incident on each $S$-vertex is equal to the label on that vertex, and such that the sum of the labels on the edges incident on each $S^+$-vertex is equal in magnitude to the label on the vertex, and equal or opposite in sign according as $d = 3$ or $d = 2$. If $\text{ADJ}(A(.),S)$ is a tree then the label on each edge $e$ is uniquely determined and has a ‘topological meaning’. Indeed, if $\text{ADJ}(A(.),S)$ is a tree and $G_1$ and $G_2$ are the two components of $\text{ADJ}(A(.),S) \setminus \{e\}$ then the label on each edge $e$ is equal to $X(A(U),U) = X(A(V),U)$, where $U = U \cup \text{COMPT}(x) \mid x$ is a vertex of $G_1$, $V = U \cup \text{COMPT}(y) \mid y$ is a vertex of $G_2$.

$(X(A(U),U) = X(A(V),U)$ because $(A(.),.)$ is $X$-regular and $U = V^C.$

*The approach followed in this paragraph was suggested by Dr Roscoe; it improves on the treatment in an earlier draft.

The labelled adjacency graph does not contain complete information about the topological structure of a digital picture, because it is unaffected by ‘knotting and linking’. Suppose, for example, that $T_0$, $T_1$ and $T_2$ are polyhedral solid tori such that $T_1 \subseteq T_0$ and $T_2 \subseteq T_0$. 

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but \( T_1 \) 'encircles the hole' of \( T_0 \) whereas \( T_2 \) does not. Then the labelled adjacency graphs corresponding to \( T_0 \setminus \text{int}(T_1) \) and \( T_0 \setminus \text{int}(T_2) \) are the same, although \( T_0 \setminus \text{int}(T_1) \) is not homotopy equivalent to \( T_0 \setminus \text{int}(T_2) \). (Example due to Dr. Roscoe.)

We now turn to the problem of proving that the familiar families of digital pictures are indeed \( X \)-regular. First, we need another definition.

**Definition**

A family \((A(\ldots))\) is \( C^*\)-regular if every component \( P \) of \( C^*(A(S),S) \) and every component \( Q \) of \( C^*(A(S),S) \) satisfies the following conditions:

(a) \( P = C^*(A(S),P \cap \mathbb{Z}_n^d) = C^*(A(P \cap \mathbb{Z}_n^d),P \cap \mathbb{Z}_n^d) \)

(b) \( Q = C^*(A(S),Q \cap \mathbb{Z}_n^d) = C^*(A(Q \cap \mathbb{Z}_n^d),Q \cap \mathbb{Z}_n^d) \)

**Proposition 3**

When \( d = 2 \) all consistent families of strongly normal digital pictures are \( C^*\)-regular and \( X \)-regular. When \( d = 3 \), \((a,b,\ldots)\) is \( C^*\)-regular and \( X \)-regular if \((a,b) = (6,26), (6,18), (26,6) \) or \((18,6)\).

**Proof (sketch)**

In the case \( d = 2 \) let \((A(\ldots))\) be any strongly normal family of digital pictures. In the case \( d = 3 \) let \((A(\ldots))\) be \((a,b,\ldots)\) where \((a,b) = (6,26), (6,18), (26,6) \) or \((18,6)\). The first (and somewhat tedious) step in the proof is to confirm that \((A(\ldots))\) is \( C^*\)-regular.

Condition (a) in the definition of \( C^*\)-regular is implied by the following assertion: suppose \( K \) is any unit cell, suppose \( P \) is a component of \( C^*(A(S),S) \). Then:

\[
(a') \quad P \cap K = C^*(A(S),P \cap \mathbb{Z}_n^d) \cap K = C^*(A(P \cap \mathbb{Z}_n^d),P \cap \mathbb{Z}_n^d) \cap K.
\]

Similarly, condition (b) in the definition of \( C^*\)-regular is implied by the following assertion.
suppose K is any unit cell, and suppose O is a component of $C^*(A(S),S)^c$. Then:

$$(b') \quad Q^\circ N_K = C^*(A(S),Q^\circ N_n \cap N_K) = C^*(A(Q^\circ N_n \cap N_K),Q^\circ N_n \cap N_K).$$

Since $(A(\cdot),\cdot)$ is consistent, $A(S)\cap K = A(S\cap N_K)\cap K$ for all unit cells $K$. Since $P$ is a component of $C^*(\Lambda,\Lambda)$, $P\cap N_K$ is a union of components of $C^*(A(S),S)\cap N_K = C^*(A(S),S\cap N_K)$, therefore $Q^\circ N_n \cap N_K$ is a union of components of $S\cap N_K$ (since $C^*$ is a continuous analogue). Since $O$ is a component of $C^*(A(S),S)^c$, $Q\cap N_K$ is a union of components of $C^*(A(S),S)^c\cap N_K$; therefore $Q^\circ N_n \cap N_K$ is a union of components of $S\cap N_K$. Thus $(a')$ is true when $S\cap N_K$ is $A(S)$-connected; for in this case $C^*(A(S),S\cap N_K)$ is connected (since $C^*$ is a continuous analogue), which implies that either $P\cap N_K = \emptyset$, or $P\cap N_K = C^*(A(S),S\cap N_K)$, and since $S\cap N_K$ is $A(S)$-connected either $Q^\circ N_n \cap N_K$ is empty, or $Q^\circ N_n \cap N_K$ is $S\cap N_K$. Similarly, $(b')$ is true when $S\cap N_K$ is $A(S)$-connected; for in this case $C^*(A(S),S)^c\cap N_K$ is connected, which implies that either $Q^\circ N_n \cap N_K = \emptyset$, or $Q^\circ N_n \cap N_K = C^*(A(S),S\cap N_K)$, and since $S\cap N_K$ is $A(S)$-connected either $Q^\circ N_n \cap N_K$ is empty, or $Q^\circ N_n \cap N_K$ is $S\cap N_K$.

So we see at once that $(a')$ holds if $(A(\cdot),\cdot) = (2S,6\cdot)$, for then $S\cap N_K$ is $A(S)$-connected, and $(b')$ holds if $(A(\cdot),\cdot) = (6,2S)$, for then $S\cap N_K$ is $A(S)$-connected.

It is also easily seen that $(a')$ holds if $(A(\cdot),\cdot) = (18,6\cdot)$, for then either $S\cap N_K$ is $A$-connected, or $(S,K)$ is of type $S$; in the latter case $(a')$ is valid by inspection.

Similarly, it is clear that $(b')$ holds if $(A(\cdot),\cdot) = (6,18\cdot)$, for then either $S\cap N_K$ is connected or $(S,K)$ is of type $18$. Also, $(a')$ is plainly valid for the fourteen types of unit cell for which $S\cap N_K$ is $6$-connected, and analogously $(b')$ holds for the fourteen types of unit cell for which $S\cap N_K$ is $6$-connected. After checking the other cases one sees that all the digital pictures under consideration are $C^*$-regular. We are only interested in the $A(S)\cap K$. So in the case $d = 2$, we only have a finite number -- indeed, a very small number -- of cases to check. In the case $d = 3$ there are just $(22 - 14)4 = 32$ cases to check.

Condition $(e)$ of $C^*$-regularity implies $X(P) = X(A(S),P\cap N^c) = X(A(P\cap N^c),P\cap N^c)$.

Moreover, we see from Table 1 that, for all unit cells $K$ and all $T \leq 2^n$,
\[ X(C^*(A(T),T^c);K) = X(C^*(A(T),T);K) \text{ if } d = 2 \text{ and } (A(\ldots)) \text{ is strongly normal, while} \]
\[ X(C^*(A(T),T^c);K) = X(C^*(A(T),T);K) \text{ if } d = 3 \text{ and } (A(\ldots)) = (6,28,\ldots), (6,18,\ldots), \]
\((18,6,\ldots), \text{ or } (26,6,\ldots). \text{ Hence (on putting } T = 0^nZ^*_N) \text{ we deduce that} \]
\[ X(A(0^nZ^*_N),0^nZ^*_N) = X(A(0^nZ^*_N),0^nZ^*_N) = X(A(0^nZ^*_N),0^nZ^*_N) \text{ in the respective cases. So, by condition (b) of } C^*-\text{regularity,} \]
\[ X(Q^c) = X(A(0^nZ^*_N),0^nZ^*_N) \text{ or} \]
\[ -X(A(0^nZ^*_N),0^nZ^*_N) \text{ according as } d = 3 \text{ or } d = 2. \]

Now let \( A'(S) \) denote the adjacency relation \( A(S^c) \). Then the family \((A'(\ldots))\) also satisfies the hypotheses of this proposition, so we have also shown that \((A'(\ldots))\) is \( C^*-\text{regular.} \) Hence, by a symmetrical argument to that just given, if \( Q' \) is a component of \( C^*(A'(S),S) \) and \( P' \) is a component of \( C^*(A'(S),S) \) then \( X(Q') = X(A'(S),Q'^c) = X(A'(S),P'^c) = X(A'(S),P'^c) \) according as \( d = 3 \) or \( d = 2. \) So on replacing \( A'(S), A'(P'^c) \) and \( A'(Q'^c) \) by \( A(S), A(P'^c) \) and \( A(Q'^c) \) respectively, we see that \((A(\ldots))\) is \( X-\text{regular.} \)

**Simple Points**

The concept of a simple point is due to Rosenfeld [Rosenfeld70] (in the two-dimensional case) and Toulaki and Mylopoulos [ToulakioMylopoulos73]. The present treatment of three-dimensional simple points is more general than that of previous workers - the paper by Toulaki and Mylopoulos deals only with the family \((6,28,\ldots)\), while the papers by Rosenfeld and Morgenthaler consider \((26,6,\ldots)\) and \((6,26,\ldots)\). Our treatment admits many other adjacency relations, in particular \((18,6,\ldots), (6,18,\ldots)\). Simple points were originally called "deletable points"; the name "simple point" seems to have been first used in the present sense in [Rosenfeld75] and, in the three-dimensional case, [Morgenthaler81]. Morgenthaler's original definition used the 'number of holes' rather than the Euler characteristic. But since the number of holes was defined by Morgenthaler in terms of the Euler characteristic there is no essential difference between his definition and ours. Tsoo
and Fu were the first to state the definition in terms of $X$ as we do here [TsaFu92].
In their original paper Toulakis and Mylopoulos defined a deletable point (=simple point) in
a different way, but we show in the next chapter that their definition is equivalent to
ours.

**DEFINITION**
Suppose $p \in S \subseteq Z^n$, and let $T$ denote the set $S \backslash \{p\}$. Let $(A(\cdot), \cdot)$ be a consistent and
$X$-regular family of digital pictures. Then $p$ is called a simple point of $(A(\cdot), S)$ if
it satisfies all the following conditions:

(i) $S\cap N(p)$ has exactly the same number of $A(S)$-components as $T\cap N(p)$ has
$A(T)$-components.

(ii) $S\cap N(p)$ has exactly the same number of $A(S)$-components as $T\cap N(p)$ has
$A(T)$-components.

(iii) $X(A(S), S\cap N(p)) = X(A(T), T\cap N(p))$

(iv) $X(A(S), S\cap N(p)) = X(A(T), T\cap N(p))$

The defining conditions for a simple point are not independent:

**PROPOSITION 4**
Suppose $p \in S \subseteq Z^n$, and let $T$ denote the set $S \backslash \{p\}$. Let $(A(\cdot), \cdot)$ be a consistent and
$X$-regular family of digital pictures. Then the following relationships hold among the
above four conditions:

(a) Conditions (iii) and (iv) are equivalent.

(b) In the case $d = 2$, if $p$ is $A(S)$-adjacent to $T$ and $A(T)$-adjacent to $S$, then the
four conditions are equivalent.

**PROOF**
(a) By symmetry it suffices to show that (iii) implies (iv). Suppose (iii) holds. Let
\( W = S(U(2^3N(p))) \). Thus \( W^2 = S^2N(p) \) and \( (W\setminus(p))^2 = T^2N(p) \). Since \((A,.) \) is consistent, \( A(W) \Delta N(p) = A(S) \Delta N(p) \) and \( A(W\setminus(p)) \Delta N(p) = A(T) \Delta N(p) \). Now

\[
\sum \{ X(C^*(A(W),W);k) \mid k \in N(p) \} \\
= \sum \{ X(C^*(A(S),S);k) \mid k \in N(p) \} \quad \text{since } A(W) \Delta N(p) = A(S) \Delta N(p) \\
= X(A(S),SN(p)) = \sum \{ X(C^*(A(S),SN(p));k) \mid k \in R_n^\circ(p) \} \text{ and } k \text{ meets } N(p) \} (\ast) \]

and similarly,

\[
\sum \{ X(C^*(A(W\setminus(p)),W\setminus(p));k) \mid k \in N(p) \} \\
= \sum \{ X(C^*(A(T),T);k) \mid k \in N(p) \} \quad \text{since } A(W\setminus(p)) \Delta N(p) = A(T) \Delta N(p) \\
= X(A(T),TN(p)) = \sum \{ X(C^*(A(T),TN(p));k) \mid k \in R_n^\circ(p) \} \text{ and } k \text{ meets } N(p) \} (\ast\ast) \]

The second term on the right-hand side of (\ast) is equal to the second term on the right-hand side of (\ast\ast), because \( A(S) \Delta (R_n^\circ(p)) = A(T) \Delta (R_n^\circ(p)) \). The first term on the right-hand side of (\ast) is equal to the first term on the right-hand side of (\ast\ast) by condition \( \text{iii} \).

So \( \sum \{ X(C^*(A(W),W);k) \mid k \in N(p) \} = \sum \{ X(C^*(A(W\setminus(p)),W\setminus(p));k) \mid k \in N(p) \} \). Also,

\[
\sum \{ X(C^*(A(W),W);k) \mid k \in R_n^\circ(p) \} = \sum \{ X(C^*(A(W\setminus(p)),W\setminus(p));k) \mid k \in R_n^\circ(p) \} \text{, since the terms in the two sums are equal (because } A(W) \Delta N(p) \setminus(p) = A(W\setminus(p)) \Delta N(p) \setminus(p)) \text{.}
\]

Therefore \( X(A(W),W) = X(A(W\setminus(p)),W\setminus(p)) \). Hence, by Proposition 2(iii), \( X(A(W),W^2) = X(A(W\setminus(p)),W\setminus(p))^2 \). Therefore \( X(A(S),SN(p)) = X(A(W),SN(p)) = X(A(W),W^2) = X(A(W\setminus(p)),W\setminus(p))^2 = X(A(T),W\setminus(p))^2 = X(A(T),TN(p)) \). So condition \( \text{iv} \) holds.

(iii) Suppose \( p \) is \( A(S) \)-adjacent to a point in \( S \), and \( A(S\setminus(p)) \) adjacent to a point in \( S^c \).

If \( (A,V) \) is a two-dimensional boxed digital-picture then \( X(A,V) \) is just one greater than the number of \( A \)-components of \( V \) minus the number of \( A \)-components of \( V^c \). Plainly

\( (SN(p))^c \) is \( A(S) \)-connected (in fact it is 4-connected), and since \( p \) is \( A(S\setminus(p)) \)-adjacent to a point in \( S \), \( (SN(p))\setminus(p))^c \) is \( A(S\setminus(p)) \)-connected. Thus \( X(A(S),SN(p)) \) is just the.
number of \( A(S) \)-components of \( S\cap N(p) \), while \( X(A(S)\backslash\{p\}, S\cap N(p)\backslash\{p\}) \) is just the number of \( A(S)\backslash\{p\} \)-components of \( S\cap N(p)\backslash\{p\} \). Hence (i) and (iii) are equivalent. Again, 
\((S\cap N(p))\backslash\{p\}\) is \( A(S)\backslash\{p\} \)-connected, since \( p \) is \( A(S)\backslash\{p\} \)-adjacent to a point in \( S^c \); and 
\((S\cap N(p))\) is \( A(S) \)-connected since \( p \) is \( A(S) \)-adjacent to a point in \( S \). So 
\( X(A(S), S\cap N(p)) \) is just the number of \( A(S) \)-components of \( S\cap N(p) \), while 
\( X(A(S)\backslash\{p\}, S\cap N(p)\backslash\{p\}) \) is just the number of \( A(S)\backslash\{p\} \)-components of \( S\cap N(p)\backslash\{p\} \). So 
(i) and (iv) are equivalent. 

COROLLARY

In the case \( d = 2 \), conditions (i) and (ii) together imply conditions (iii) and (iv).

PROOF

If conditions (i) and (ii) both hold, then \( p \) is \( A(S) \)-adjacent to \( T \) and \( A(T) \)-adjacent to \( S^c \). So the corollary follows from part (b) of the above proposition.

REMARK

In the case \( d = 3 \), conditions (i), (ii) and (iii) are in general independent, but we shall show in the next chapter that when \( (A(.)..) = (6,26,..) \) or \( (6,18,..) \) (ii) and (iii) together imply (i), and when \( (A(.)..) = (26,6,..) \) or \( (18,6,..) \) (i) and (iii) together imply (ii).

There is a rather attractive characterization of simple points:

PROPOSITION 5

Let \( (A(.)..) \) be a consistent and \( X \)-regular family of digital pictures, and let \( S \) be a 
subset of \( Z^d \). Then \( p \) is a simple point of \( (A(.).., S) \) if and only if 
\( \text{ADJ}(A(.).., N(p), S\backslash\{p\}) = \text{ADJ}(A(.).., N(p), S) \).

PROOF

It is plain that if \( \text{ADJ}(A(.).., N(p), S\backslash\{p\}) = \text{ADJ}(A(.).., N(p), S) \) then \( p \) is simple. To prove
the converse, suppose $p$ is simple. We first show that $\text{adj}(A(S), N(p), S) = \text{adj}(A(T), N(p), T)$.

Suppose $B$ is an $A(S)$-component of $S^0 N(p)$ and $C$ is an $A(T)$-component of $T^0 N(p)$. Then, since $(A(S), A(T))$ is consistent, (i) implies that $B \setminus \{p\}$ is an $A(S)$-component of $S^0 N(p)$. While (ii) implies that $C \setminus \{p\}$ is an $A(S)$-component of $S^0 N(p)$. Thus the map $X \mapsto X \setminus \{p\}$ defines a bijection of the $A(S)$-components of $S^0 N(p)$ onto the $A(S)$-components of $S^0 N(p)$, and also defines a bijection of the $A(S)$-components of $S^0 N(p)$ onto the $A(T)$-components of $T^0 N(p)$. We now claim that $C \setminus \{p\}$ is $A(S)$-adjacent to $B$ if and only if $C$ is $A(T)$-adjacent to $B \setminus \{p\}$. We shall prove this in the case $d = 3$ -- in the case $d = 2$, the same method of proof works, but we must replace $6$-adjacent with $4$-adjacent throughout. So suppose $d = 3$. Then by Proposition $1(iv)$ in Chapter $2$ it suffices to prove that $C \setminus \{p\}$ is $6$-adjacent to $B$ if and only if $C$ is $6$-adjacent to $B \setminus \{p\}$.

Suppose, for the purpose of getting a contradiction, that $C \setminus \{p\}$ is $6$-adjacent to $B$ but $C$ is not $6$-adjacent to $B \setminus \{p\}$. Then $p \in B$ and $C \setminus \{p\}$ is $6$-adjacent to $p$, so that $p \notin C$ (because $C$ is an $A(T)$-component). Let $E$ denote the set whose members are the twelve $18$-neighbours of $p$ that are not $6$-adjacent to $p$. Now $B$ is an $A(S)$-component of $S^0 N(p)$, and $C$ is an $A(T)$-component of $T^0 N(p)$. So since $p \in C$, and (by hypothesis) $C$ is not $6$-adjacent to $B \setminus \{p\}$, all the $6$-neighbours of $p$ are in $S^C$ and must therefore belong to $C$. Every point in $E$ is either a $6$-neighbour of $p$ or is $6$-adjacent to a $6$-neighbour of $p$. So every point in $E$ is either in $C$ or is $6$-adjacent to a point in $C$. Hence $E \cap S^C \subseteq C$ and, since $C$ is not $6$-adjacent to $B \setminus \{p\}$, $E \cap B = \emptyset$. In other words, $E \subseteq C \cup (S \setminus B)$. Neither $C$ nor $S \setminus B$ can be $6$-adjacent to $B$, so $E$ is not $6$-adjacent to $B$. This implies that $p$ has no $26$-neighbours in $B$, whence $B = \{p\}$. Thus $p$ does not satisfy condition (i) in the definition of a simple point. This contradiction proves that the "only if" part of our claim holds.

The "if" part is proved by deriving an analogous violation of condition (ii) of the definition of a simple point; the argument is the same, but with $B$'s and $S$'s interchanged with $C$'s and $T$'s.

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We have now shown that \( \text{adj}(A(S),N(p),S) = \text{adj}(A(T),N(p),T) \). By the Corollary to Proposition 4, this proves Proposition 5 in the case \( d = 2 \). In the case \( d = 3 \), let \( R_0 \) and \( U_0 \) be respectively the \( A(S) \)-component of \( S^N(p) \) that contains \( p \) and the \( A(T) \)-component of \( T^N(p) \) that contains \( p \). Then (as we observed above) \( R_0 \setminus \{ p \} \) is an \( A(T) \)-component of \( T^N(p) \) and \( U_0 \setminus \{ p \} \) is an \( A(S) \)-component of \( S^N(p) \). All other \( A(S) \)-components of \( S^N(p) \) and \( A(T) \)-components of \( T^N(p) \) are respectively \( A(T) \)-components of \( T^N(p) \) and \( A(S) \)-components of \( S^N(p) \), and conversely. So to prove that \( \text{adj}(A(.),N(p),S) = \text{adj}(A(.),N(p),T) \) we need only show \( X(A(R_0),R_0) = X(A(R_0 \setminus \{ p \}),R_0 \setminus \{ p \}) \) and \( X(A(U_0),U_0) = X(A(U_0 \setminus \{ p \}),U_0 \setminus \{ p \}) \).

We assert that
\[
X(A(S),S^N(p)) = X(C^*(A(S),S^N(p))) = X(C^*(A(S^N(p)),S^N(p))) = X(A(S^N(p)),S^N(p)) = X(A(R_0),R_0) + \sum X(A(R),R) | R \text{ is an } A(S^N(p)) \text{-component of } S^N(p) \text{ and } R \neq R_0
\]

Since \( A(\ldots) \) is \( X \)-regular, \( (A(S),S) \) is strongly normal, which implies that \( (A(S),S^N(p)) \) is strongly normal. Thus the first and third equalities are just the definitions of \( X(A(S),S^N(p)) \) and \( X(A(S^N(p)),S^N(p)) \). The second equality holds because \( A(S) = A(S^N(p)) \) in \( N(p) \) (since \( A(\ldots) \) is consistent), which implies that \( C^*(A(S),S^N(p)) = C^*(A(S^N(p)),S^N(p)) \). The last equality follows from Proposition 2(ii).

By similar reasoning we deduce that
\[
X(A(T),T^N(p)) = X(A(R_0 \setminus \{ p \}),R_0 \setminus \{ p \}) + \sum X(A(R),R) | R \text{ is an } A(T^N(p)) \text{-component of } T^N(p) \text{ and } R \neq R_0 \setminus \{ p \}
\]

Now condition (iii) in the definition of a simple point implies that the left-hand sides of (S) and (S$\S$) are equal, and the \( \Sigma \)-term on the right-hand side of (S) is equal to the \( \Sigma \)-term on the right-hand side of (S$\S$), because the terms in the \( \Sigma \)'s are equal. Hence the first term on the right-hand side of (S) is equal to the first term on the right-hand
side of (88). It follows that \( X(A(R_o), R_o) = X(A(R_o \setminus \{p\}), R_o \setminus \{p\}) \), as required.

For all \( V \in \mathbb{Z}_n^3 \) let \( A'(V) \) denote the adjacency relation \( A(V^c) \). It is readily seen that the defining conditions of a simple point of \((A(\cdot), S)\) are just the same as the defining conditions of a simple point of \((A'(\cdot), T^c)\). Hence \( p \) is a simple point of \((A'(\cdot), T^c)\).

Also, \( U_o \) is the \( A'(T^c) \)-component of \( S^o N(p) \) that contains \( p \), and \( U_o \setminus \{p\} \) is an \( A'(S^c) \)-component of \( S^o N(p) \). So by a symmetrical argument to the above we deduce that \( X(A'(U_o), U_o) = X(A'(U_o \setminus \{p\}), U_o \setminus \{p\}) \), which is equivalent to \( X(A(U_o^c), U_o) = X(A(U_o \setminus \{p\})^c, U_o \setminus \{p\}) \). ■

* The argument in this paragraph is due to Dr. Roscoe, who pointed out a mistake in my original argument.
Chapter 4 - A GENERAL THEORY OF IMAGE SHRINKING

Introduction

This chapter develops a general theory of image shrinking for two- and three-dimensional digital pictures. As we mentioned in Chapter 1, there is a need for a precise definition of "topology preservation" for three-dimensional shrinking and thinning algorithms. Our approach is to define shrinking for polyhedra, and then to say that one digital picture is a shrunken image of another if the standard analogue of the first can be obtained by shrinking the standard analogue of the second. It is clear from Chapter 3 that this definition is applicable to digital pictures based on a wide variety of adjacency relations.

Let \( p \) be an object point in a digital picture. Suppose we change \( p \) into a background point. Under what circumstances will this produce a shrunken image of the original picture? The main theorems in this chapter assert that for all of the commonly used adjacency relations this will be so when and only when \( p \) is a simple point of the original picture.

In [TouleakisMyopoulos79] Touleakis and Myopoulos introduced the notion of a deletable point for digital pictures based on the \((4,8)\), \((6,26)\) and analogous higher dimensional adjacency relations. In the two-dimensional case their definition is equivalent to that used by earlier authors (e.g. [Rosenfeld70]). We show that in the three-dimensional case their deletable points are precisely the \((6,26)\) simple points.

Elementary Terminology

A subcomplex of a complex \( K \) is a subset of \( K \) which is itself a complex. If \( K \) and \( L \) are complexes such that \( |K| = |L| \) and every simplex in \( K \) is contained in some simplex in \( L \) then we say that \( K \) is a subdivision of \( L \) and write \( K \succ L \). The dimension of a complex is the maximum of the dimensions of its simplexes. If \( C \) is a simplex then we call \( \partial C \) the relative boundary of \( C \). The relative interior of a simplex \( C \) is the set \( C \setminus \partial C \). If
K is a complex and n is any non-negative integer then $K^n$ denotes the n-skeleton of K, which is the set of all the simplexes in K whose dimension is at most n.

For any complex L, and any point $p$, $N_L(p)$ denotes the simplicial neighborhood of p in L, which is defined to be the minimal complex that contains all the simplexes of L that contain p. For any complex L, and any point $p$, $L_k(p)$ denotes the link of p in L, which is defined to be the subcomplex of $N_L(p)$ consisting of all the simplexes of $N_L(p)$ that do not contain p.

Let $\sigma$ be an m-simplex of the complex K and let $\tau$ be an (m-1)-dimensional face of $\sigma$. We call the pair ($\sigma$, $\tau$) an m-dimensional free pair of simplexes in K if $\tau$ is not a face of any other simplex of K.

**A Precise Definition of Shrinking for Polyhedra**

**DEFINITION**

Let $P$ be any polyhedron, and let $K$ be a complex such that $|K| = P$. Let $(C,F)$ be a free pair of simplexes in K and let $L$ be the complex $K \setminus (C,F)$. Then we define

$\text{COLLAPSE}(P,C,F) = |L|$. 

Observe that $\text{COLLAPSE}(P,C,F)$ does not actually depend on the complex K, and that it is easy to give an alternative definition which does not make use of K:

**PROPOSITION 0**

Let $P$ be any polyhedron and, for some integer $n \leq d$, let $C$ and $F$ be respectively an $n$-simplex and an $(n-1)$-simplex contained in $P$ such that the following conditions are satisfied:

(i) $C \setminus \text{Bd } C$ does not meet $\text{cl}(P \setminus C)$ (this condition is vacuous if $n = d$)

(ii) $F$ is a face of $C$ and $F \setminus \text{Bd } F$ does not meet $\text{cl}(P \setminus C)$

Then $\text{COLLAPSE}(P,C,F)$ is defined, and is equal to $(P \setminus C) \cup ((\text{Bd } C) \setminus F) \cup \text{Bd } F$. 

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Proof
This is straightforward and is left to the reader.

DEFINITION
Suppose \( P_0 \supseteq P_1 \supseteq \ldots \supseteq P_n (n \geq 0) \) is a sequence of polyhedra such that for all \( 0 \leq i < n \) there exist simplexes \( C_i \) and \( F_i \) such that \( P_{i+1} = \text{COLLAPSE}(P_i, C_i, F_i) \). Then we say that \( P_n \) is a shrunken image of \( P_0 \).

Observe that the relation "is a shrunken image of" is a partial order on the set of all polyhedra. In the terminology of [Hudson69], \( Q \) is a shrunken image of \( P \) if and only if \( P \) collapses to \( Q \) ([Hudson69], Thm 2.4).

It is easy to see that if \( M \) is a complex in \( \mathbb{R}^d \) such that \( |M| \) is boxed then there is a \((d-1)\)-dimensional subcomplex \( N \) of \( M \) such that \( |N| \) is a shrunken image of \( |M| \). (\( N \) is obtained by starting with \( M \) and repeatedly removing \( d \)-dimensional free pairs of simplexes until none remain.)

When \( d = 2 \), the relation "is a shrunken image of" can be characterized without using the notion of collapsing. This is demonstrated in Theorem 1 below. But there seems to be no easy way of doing this in the case \( d = 3 \). Thus one might attempt to characterize "\( Q \) is a shrunken image of \( P \)" by means of condition (\( ii \)) in Theorem 1 and the additional requirement that the inclusion map \( i : Q \to P \) should induce a 1-1 correspondence between the "holes" in \( P \) and the "holes" in \( Q \). This requirement can be stated precisely: each closed curve in \( P \) should be continuously deformable in \( P \) to a closed curve in \( Q \), and any closed curve in \( Q \) that is continuously deformable in \( P \) to a single point should also be continuously deformable in \( Q \) to a single point (c.f. [Margenthaler81]). However, these conditions do not imply that \( Q \) is a shrunken image of \( P \), as Hudson gives an example of a polyhedron \( P \) (which can be embedded in \( \mathbb{R}^3 \)) which satisfies the stated conditions whenever \( Q = \langle x \rangle \) and \( x \in \mathcal{P} \), such that the only shrunken image of \( \mathcal{F} \) is \( P \) itself. ([Hudson69], Ch II, Section 1.)
FIGURE 1: Morgenthaler's Counterexample
THEOREM 1

Suppose $X$, $P$ and $Q$ are polyhedra such that $Q \subseteq P \subseteq X$, and $P \setminus Q \subseteq \text{int}(X)$. Consider the following three conditions:

(i) $Q$ is a shrunken image of $P$

(ii) Each component of $P$ meets $Q$ in a component of $Q$, and each component of $X \setminus Q$ meets $X \setminus P$ in a component of $X \setminus P$

(iii) Suppose $U$, $V$, $U'$ and $V'$ are components of $P$, $X \setminus Q$, $Q$ and $X \setminus P$ respectively, where $U' = U \cap Q$ and $V' = V \cap P$. Then $U$ meets $\text{cl}(V')$ if and only if $U'$ meets $\text{cl}(V)$. Furthermore, $X(U) = X(U')$ and $X(X \setminus U) = X(X \setminus U')$

If $d = 2$ then (i) and (ii) are equivalent and imply (iii). If $d = 3$ then (i) implies (ii) and (iii), but (i) may be false even if (ii) and (iii) are both true.

PROOF

To show that (i) implies (ii) and (iii) it suffices to prove that (ii) and (iii) hold when $Q = \text{COLLAPSE}(P,C,F)$. It is intuitively clear that this is so, and it is a straightforward matter to give a rigorous proof. Such a proof would, however, be rather uninteresting and is therefore omitted.

Properties (ii) and (iii) do not imply (i) when $N \neq 3$

In the case $d = 3$, Morgenthaler's counterexample (see Figure 1) shows that (i) may fail even if (ii) and (iii) hold.

(iii) implies (i) when $N \neq 2$

We shall assume that $X$ is connected; this involves no loss of generality, since if $X$ is not connected then we can consider one component of $X$ at a time.

Suppose $d = 2$ and $P$ and $Q$ satisfy (ii). Let $T_1 \subseteq T \subseteq K$ be simplicial complexes such that $|K| = X$, $|T| = P$ and $|T_1| = Q$. For any subcomplex $W$ of $K$ such that $T_1 \subseteq W \subseteq T$, let
\( S(W), F(W) \) be a free pair of simplexes in \( W \) such that neither \( S(W) \) nor \( F(W) \) belong to \( T_i \), provided that such a free pair of simplexes exists.

Define \( W_0 = T_i \). As long as there exist \( F(W_i) \) and \( S(W_i) \) satisfying the above definition, define \( W_{i+1} = W_i \setminus (S(W_i), F(W_i)) \). It is easily shown by induction on \( i \) that if \( W_i \) exists then it is a subcomplex of \( K \) such that \( 1, \omega W_0, \omega K \). Note in particular that if \( W_i \) exists then \( W_i \) is a \( T_i \), so that \( |W_i| \geq 0 \). Now \( |W_0| = P \) and \( |W_{i+1}| = \text{COLLAPSE}(|W_i|, S(W_i), F(W_i)) \). So, for all \( i \) for which \( W_i \) is defined, \( |W_i| \) is a shrunken image of \( P \). To finish the proof, we have to show that there is some \( j \) such that \( |W_j| = 0 \), or, equivalently, \( W_j = T_i \).

Since \( |W_i| \) (if it exists) is a shrunken image of \( P \), on applying the \'(i) implies \'(ii)\' part of this theorem to the set \( |W_i| \), we deduce that each component of \( P \) meets \( |W_i| \) in a component of \( |W_i| \), and each component of \( X \setminus |W_i| \) meets \( X \setminus P \) in a component of \( X \setminus P \). (*)

For all \( j \geq 0 \) such that \( F(W_j) \) exists \( W_{i+1} \) is defined and is strictly contained in \( W_j \).

Since \( W_0 = T_i \) is a finite set, this sequence must eventually terminate. Hence there is some \( m \) for which there does not exist \( F(W_m) \) satisfying the above definition. We shall show now that \( W_m = T_i \).

Suppose first of all that \( W_m \setminus T_i \) contains a 2-simplex. Let \( Y \) be the set of all 2-simplexes in \( W_m \setminus T_i \), and let \( Y' \) denote the set of all "free edges" of \( Y \) — that is, the set of all 1-simplexes that are sides of just one 2-simplex in \( Y \). Then \( \text{int}(UY) \subseteq X \setminus Q \) and \( UY' = \text{Bd}(UY) \). Since \( Y \) is (by hypothesis) non-empty and \( UY \) is bounded, the set \( \text{Bd}(UY) \), and hence the set \( Y' \), is non-empty. Now for all \( e \in Y' \), \( e \) is a side of a 2-simplex in \( W_m \setminus Y = T_i \) (for otherwise \( e \) satisfies the conditions for \( F(W_m) \), contrary to the definition of \( m \)). Hence \( Y' \subseteq T_i \); thus \( \text{Bd}(UY) = UY' \subseteq Q \). Now pick a point \( r \) in the interior of a 2-simplex in \( Y \) and let \( R \) be the component of \( X \setminus Q \) that contains \( r \). As \( \text{Bd}(UY) \subseteq Q \), the component \( R \) cannot meet \( \text{Hd}(UY) \). Therefore \( R \subseteq \text{int}(UY) \subseteq UY \subseteq P \), and so \( R \) does not meet \( X \setminus P \), contrary to \'(ii)\'. This contradiction proves that \( W_m \setminus T_i \) contains no 2-simplexes.
Let $G$ be the smallest graph containing $W_\alpha \setminus T_1$. Thus the edges of $G$ are the 1-simplices in $W_\alpha \setminus T_1$ and the vertices of $G$ are the isolated points in $|W_\alpha \setminus T_1|$ and the endpoints of all the 1-simplices in $W_\alpha \setminus T_1$. Note that if $v$ is a vertex of $G$ then apparently $\{v\}$ may or may not belong to $T_1$; but no edge of $G$ can belong to $T_1$.

Now suppose $G$ contains a cycle $\Gamma$. The point-set $U \Gamma$ is a simple closed polygonal curve. Pick a point $w$ on $U \Gamma$ such that $w$ is not a vertex of $\Gamma$. Then $w \in P \setminus Q = \text{int}(X)$, so by the Jordan Curve Theorem $w$ is on the boundary of two different components $C_1$, $C_2$ of $X \setminus U \Gamma$. By definition of $\Gamma$, no edge of the cycle $\Gamma$ is a side of a 2-simplex in $T_1$, and so (since $W_\alpha \setminus T_1$ contains no 2-simplices) no edge of $\Gamma$ is a side of a 2-simplex in $W_\alpha$. Hence $w \in \text{cl}(\{X \setminus W_\alpha\} \cap C_1)$ and $w \in \text{cl}(\{X \setminus W_\alpha\} \cap C_2)$. Since $w \in X \setminus Q$, the (non-empty) sets $(X \setminus W_\alpha) \cap C_1$ and $(X \setminus W_\alpha) \cap C_2$ are contained in the same component of $X \setminus Q$. But they are contained in two distinct components of $X \setminus W_\alpha$, and by $(\ast)$ each of these components of $X \setminus W_\alpha$ contains a component of $X \setminus P$. Thus we have found a component of $X \setminus Q$ that contains two different components of $X \setminus P$. This contradiction to (ii) proves that $G$ is an acyclic graph (i.e., a forest).

Now suppose $G$ has an edge $e$. The component of $G$ that contains $e$ is a tree, and must possess two vertices $u$ and $v$ of degree 1. Then $\{u\} \in T_1$ and $\{v\} \in T_1$, for otherwise $u$ or $v$ would satisfy the defining condition for $F(W_\alpha)$, contrary to our hypothesis that $F(W_\alpha)$ does not exist. Thus $u$ and $v$ lie in $U$. Let $y$ be the first vertex (after $u$) on the unique path from $u$ to $v$ in $G$ such that $y \in Q$. Then (ii) implies that $u$ and $y$ belong to the same component of $Q$, since by their definition they belong to the same component of $|W_\alpha|$ and hence of $P$. Let $\Pi$ be the path from $u$ to $y$ in $G$. Then the point-set $U \Pi$ is a simple polygonal arc. Join the endpoints $u$ and $y$ of $U \Pi$ with a simple polygonal arc in $Q$ from $y$ to $u$, to produce a simple closed polygonal curve $K$. Pick a point $w$ on $U \Pi$ such that $w$ is not a vertex of $G$. Then $w \in P \setminus Q = \text{int}(X)$, so by the Jordan Curve Theorem $w$ is on the boundary of two different components $C_1$, $C_2$ of $X \setminus Q$, and we derive a contradiction as in the previous paragraph. Thus $G$ has no edges.
Now suppose $G$ contains a vertex $v$ that does not lie in $Q$. Then $\{v\} \notin T_1$. Hence $\{v\}$ is a component of $|W_n|$, and so it follows from (*) that the component of $P$ that contains $v$ does not meet $|W_n| \setminus \{v\}$. Hence this component of $P$ cannot meet $Q$, which contradicts (ii). Therefore every vertex of $G$ lies in $Q$, whence (by the definition of $G$) $W_n = T_1$, as asserted.  

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A Precise Definition of Shrinking for Digital Pictures

DEFINITION

Let $(A(i),..)$ be a consistent family of digital pictures. The digital picture $(A(T),T)$ is a shrunken image of the digital picture $(A(S),S)$ if $C^*(A(T),T)$ is a shrunken image of $C^*(A(S),S)$.

THEOREM 1'

Suppose $(A(i),..)$ is a consistent and $X$-regular family of digital pictures and suppose $1 \leq S \leq Z_n^+$. Consider the following conditions:

(i) $(A(T),T)$ is a shrunken image of $(A(S),S)$

(ii) Each $A(S)$-component of $S$ meets $T$ in an $A(T)$-component of $T$. and each

\[ A(T) \text{-component of } T \]  

meets $S$ in an $A(S)$-component of $S$.

(iii) Suppose $U$ is an $A(S)$-component of $S$ and $V$ is an $A(T)$-component of $T$. Let $U' = U \cap T$ and let $V' = V \cap S$. Then $U$ is $A(S)$-adjacent to $V'$ if and only if $U'$ is $A(T)$-adjacent to $V$. Furthermore, $X(A(S),U) = X(A(T),U')$ and $X(A(T),V) = X(A(S),V')$.

If $d = 2$ then (i) and (ii) are equivalent and imply (iii). If $d = 3$ then (i) implies (ii) and (iii), but (i) may be false even if (ii) and (iii) are both true.
PROOF

For $d = 3$, Morgenthaler's counterexample shows that (i) may fail even if (ii) and (iii) both hold. Put $Q = c^*(A(T), T)$ and $P = c^*(A(S), S)$ in Theorem 1. Then, by the definition of a continuous analogue, conditions (i), (ii) and (iii) in Theorem 1 are equivalent to the corresponding conditions in Theorem 1'.

COROLLARY

If $(A(\cdot), \cdot)$ is consistent and $X$-regular, and if $(A(T), T)$ is a shrunken image of $(A(S), S)$ then $adj(A(\cdot), S) = adj(A(\cdot), T)$.

We can now show in the two-dimensional case of (8.4) adjacency our definition of shrinking is equivalent to the definition given in [Hilditch69]. It is enough to show that Hilditch's definition is equivalent to condition (ii) in Theorem 1'. First, it is plain that if for all $W \subseteq \mathbb{Z}^2$ we let $C_1(W)$ denote the union of all the unit squares with sides parallel to the coordinate axes whose centres are the points in $W$, then $C_1(S)$ is always a continuous analogue of $(8.4, S)$. So if condition (ii) holds then, by Theorem 1, $C_1(T)$ is a shrunken image of $C_1(S)$, which in turn implies that $C_1(T)$ is a "continuous deformation" of $C_1(S)$, as stipulated by Hilditch. Conversely, the most natural topological interpretation of the term "continuous deformation" in the context of Hilditch's definition is "deformation retraction", and it follows from standard results which are to be found in texts on algebraic topology (e.g. [Maunder70]) that if $C_1(T)$ is a deformation retract of $C_1(S)$ then (ii) holds.

Again, since (i) is equivalent to (ii) when $d = 2$, the criterion used in [Stefanelli; Rosenfeld71] and elsewhere to check that a thinning algorithm preserves topology is equivalent to our definition of shrinking.

The concept of a simple point plays a key role in the theory of shrinking. For the most commonly used families of digital pictures $(A(\cdot), \cdot)$, it turns out that $p$ is a simple point of $(A(\cdot), S)$ if and only if $(A(S \setminus \{p\}), S \setminus \{p\})$ is a shrunken image of $(A(S), S)$. Our main aim in the rest of this chapter will be to prove this result for $(6, 18, \cdot)$, $(18, 6, \cdot)$.  

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(6.25.3) and (25.6.3). We begin with:

**PROPOSITION 2**

Let \( (A(.), S) \) be a consistent and \( X \)-regular family of digital pictures. Suppose \( (A(S\{p}\}, S\{p}\}) \) is a shrunken image of \( (A(S), S) \), where \( p \in S \subseteq \mathbb{Z}^d \). Then \( p \) is a simple point of \( (A(.), S) \).

**PROOF**

Suppose \( (A(S\{p}\}, S\{p}\}) \) is a shrunken image of \( (A(S), S) \). Then \( C^*(A(S\{p}\}, S\{p}\}) \) is a shrunken image of \( C^*(A(S), S) \). Let \( V = (Z^* \cap N(p)) \cup G \). Then since \( (A(.), S) \) is consistent, \( A(S) \cap N(p) = A(V) \cap N(p) \). Thus each \( A(S) \)-component of \( S^* \cap N(p) \) is an \( A(V) \)-component of \( V \). We observe that \( C^*(A(V\{p}\}, V\{p}\}) \) is a shrunken image of \( C^*(A(V), V) \), because \( C^*(A(V), V) \cap N(p) \) and \( C^*(A(V\{p}\}, V\{p}\}) \cap N(p) \) are respectively equal to \( C^*(A(S), S) \cap N(p) \) and \( C^*(A(S\{p}\}, S\{p}\}) \cap N(p) \).

The fact that \( (A(.), S) \) is consistent implies that \( A(S) = A(S\{p}\} \cap Z^* \), and therefore \( C^*(A(S\{p}\}, S\{p}\}) \cap (\text{int}(N(p)))^c = C^*(A(S), S) \cap (\text{int}(N(p)))^c \). It follows that \( C^*(A(S), S) \cap C^*(A(S\{p}\}, S\{p}\}) \subseteq \text{int}(N(p)) \). Similarly \( C^*(A(V\{p}\}, V\{p}\}) \cap C^*(A(V), V) \subseteq \text{int}(N(p)) \). Since \( C^*(A(S\{p}\}, S\{p}\}) \) is a shrunken image of \( C^*(A(S), S) \), and since \( C^*(A(V\{p}\}, V\{p}\}) \) is a shrunken image of \( C^*(A(V), V) \), it follows from Theorem 1, the definition of a continuous analogue, and the fact that \( (A(.), S) \) is \( X \)-regular, that \( \text{ADJ}(A(.), N(p), S) = \text{ADJ}(A(.), N(p), S\{p}\}) \).

**COROLLARY**

If \( d = 2 \) then \( (A(S\{p}\}, S\{p}\}) \) is a shrunken image of \( (A(S), S) \) if and only if \( p \) is a simple point of \( (A(.), S) \).

**PROOF**

If \( d = 2 \) and \( p \) is a simple point of \( (A(.), S) \) then by Proposition 4 in Chapter 3 \( \text{adj}(A(S), S) = \text{adj}(A(S\{p}\}, S\{p}\}) \) and \( (A(S\{p}\}, S\{p}\}) \) is a shrunken image of
(A(S),S) by Theorem 1'. The converse follows from Proposition 2.

This corollary may fail in the case d = 3. Figure 2 shows counterexamples when the adjacency relation is (18,18), (18,26), (26,18), (26,26) or A_n. (The counterexample for A(.) = A_n is due to Dr. Ted Pappas.) Nevertheless, the converse of Proposition 2 is valid for (6,26,), (25,6,), (6,18,) and (18,5,). In the following section we prepare the ground for a proof of this.

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A(.) = (18,26,) or (18,18,)

FIGURE 2: p is a simple point of (A(.),S), but (A(S\{p\}),S\{p\}) is NOT a shrunken image of (A(S),S)

Notes to Figure 2: p and the "ones" are the points in S; the A(S)-neighbours of p are shown in italic (\& or \).
(q,T)-Consistent Families of Polyhedra

NOTATION
For all \( \epsilon > 0 \) let \( D(p,\epsilon) \) denote the interior of the solid cube whose centroid is \( p \), whose sides are of length \( \epsilon \) and are parallel to the coordinate axes.

DEFINITION
Suppose \( q \in T \subseteq \mathbb{Z}_n^3 \) and suppose \( \{P(S) \mid S \subseteq \mathbb{Z}_n^3\} \) is a family of polyhedra in \( \mathbb{R}_n^3 \) such that, for all \( S \subseteq \mathbb{Z}_n^3 \), \( P(S) \cap \mathbb{Z}_n^3 = S \). We say \( \{P(S) \mid S \subseteq \mathbb{Z}_n^3\} \) is \((q,T)\)-consistent if \( q \in P(T) \) and, for all sufficiently small \( \epsilon \), \( P(T \setminus \epsilon) \) is a shrunken image of \( P(T) \setminus D(q,\epsilon) \).

For brevity, we shall often use \( P(A(\cdot),\cdot) \) to denote a family \( \{P(A(S),S) \mid S \subseteq \mathbb{Z}_n^3\} \) of polyhedra in \( \mathbb{R}_n^3 \), indexed by a family \( \{(A(S),S) \mid S \subseteq \mathbb{Z}_n^3\} \) of digital pictures. Observe that the family \( C^*(A(\cdot),\cdot) \) certainly satisfies the condition that for all \( S \subseteq \mathbb{Z}_n^3 \), \( C^*(A(S),S) \cap \mathbb{Z}_n^3 = S \).

REMARK
It is easy to see that \( \{P(S) \mid S \subseteq \mathbb{Z}_n^3\} \) is \((q,T)\)-consistent if and only if there is a complex \( L \) such that \( |L| = P(T) \), and such that \( P(T \setminus \epsilon) \) is a shrunken image of \( (L \setminus N_L(q)) \cup L \cup L_K(q) \cup L \cup L_K(q) \), (of course, the subcomplex of \( L \) consisting of all the simplexes of \( L \) which do not have \( q \) as a vertex.)

PROPOSITION 3
(i) If \( d = 2 \), and \( A(\cdot,\cdot) \) is a consistent family of strongly normal digital pictures then \( C^*(A(\cdot,\cdot)) \) is \((q,T)\)-consistent for all \( q \in T \subseteq \mathbb{Z}_n^3 \).
(ii) If \( a = 18 \) or 26 then \( C^*(c(\cdot)S,\cdot) \) is \((q,T)\)-consistent for all \( q \in T \subseteq \mathbb{Z}_n^3 \).
(iii) \( C^*(6;18,\cdot) \) is \((q,T)\)-consistent for all \( q \in T \subseteq \mathbb{Z}_n^3 \).
(iv) \( C^*(G, 26, \ldots) \) is \((q, T)\)-consistent if and only if \( q \in T \subseteq \mathbb{Z}_n^2 \) and for each unit cell \( K \) in \( N(q) \), either all corners of \( K \) are in \( T \) or \( q \) is \( 18\)-adjacent to \( T^c \).

**Proof**

We shall use the following easy lemma, whose proof is left to the reader:

**Lemma**

Let \( X \subseteq Y \) be polyhedra in \( \mathbb{R}^n \) such that for all unit cells \( K < X < Y \), \( Y \) is a shrunken image of \( Y \), and \( K \) is the interior of \( K \). Then \( X \) is a shrunken image of \( Y \).

---

**Proof of (i)**

For brevity, let \( C(S) \) denote \( C^*(A(S), S) \).

Let \( V \) be any subset of \( \mathbb{Z}^2_n \). If \( V = \emptyset \) then \( C(V) \) has a strongly regular dissection (viz. the empty complex). We prove part (i) of the proposition by induction on the number of points in \( V \). More precisely, we claim that given any strongly regular dissection \( J \) of \( C(V) \) and any point \( p \in V \) there exists a strongly regular dissection \( L \) of \( C(V^c(p)) \) such that \( J \) is precisely the subcomplex of \( L \) consisting of all the simplexes of \( L \) that do not have \( p \) as a vertex. Indeed, \( L \) is the minimal complex such that:

(a) \( L \) contains \( J \cup \{p\} \) as a subcomplex

(b) Each straight line segment joining \( p \) to an \( A(V^c(p)) \)-neighbour of \( p \) in \( V \) is a 1-simplex of \( L \)

(c) If \( K \) is a unit cell containing \( p \) (or one corner) such that exactly two corners of \( K \) are in \( V \), and the three points in \( \{p\} \cup (K \setminus V) \) are pairwise \( A(V^c(p)) \)-adjacent, then the \((1, 1, \sqrt{2})\) triangle whose corners are the three points in \( K \cup (V \setminus \{p\}) \) is a 2-simplex of \( L \)

(d) If \( K \) is a unit cell containing \( p \) such that all other corners of \( K \) are in \( V \), and the two 6-neighbours of \( p \) in \( K \) are \( A(V^c(p)) \)-adjacent to each other, then the two \((1, 1, \sqrt{2})\) triangles whose common hypotenuse is the diagonal of \( K \) not
containing p are both $Z$-simplexes of L

(e) If K is a unit cell containing p, and every other corner of K is in V, but the
two 6-neighbours of p in K are not $A(U(V(p)))$-adjacent to each other, then the
two $(1,1,\sqrt{2})$ triangles whose common hypotenuse is the diagonal of K containing
p are both Z-simplexes of L.

Using the above lemma, it is easy to show that L has the stated properties. (Observe
that in (d) and (e) "$A(U(V(p)))$-adjacent" is the same as "$A(V)$-adjacent".)

Proof of (iii)

Suppose $p \in Z_n^3$. Define a total order $<<$ on $Z_n^3$ such that $p$ is the $<<$-least point in $Z_n^3$.

For all $S \subseteq Z_n^3$, define $G(S)$ to be the minimal complex that contains (as 1-simplexes), all
of the straight line segments whose endpoints, x and y say, satisfy one of the following
four conditions:

(a) $x$ and $y$ are 6-neighbours in $S$

(b) $x$ and $y$ are 18-neighbours in $S$ but are not 6-neighbours, and there is at most
    one point in $S$ that is 6-adjacent both to $x$ and to $y$

(c) $x$ and $y$ are 18-neighbours in $S$, there are two points $s$ and $t$ in $S$ each of which
    is 6-adjacent both to $x$ and to $y$, and one of $s$ and $t$ is the $<<$-least point of
    $\{x,y,s,t\}$

(d) one of $x$ and $y$ is the centroid of a unit cell and the other is a point in $S$ in
    that unit cell

For any subcomplex $M$ of $G(S)$, define $G_2(M)$ to be the complex whose 1-skeleton is $M$
and whose other simplexes are the $(1,1,\sqrt{2})$, $(1,\sqrt{3}/2,\sqrt{3}/2)$ and
$(\sqrt{2},\sqrt{3}/2,\sqrt{3}/2)$ triangles whose sides are 1-simplexes of $M$. Define $G_3(M)$ to be the
complex whose 2-skeleton is $G_2(M)$ and whose other simplexes are the 3-simplexes all of
whose two-dimensional faces are 2-simplexes in $G_2(M)$. 

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Define $G(S) = G_2(G_1(S))$. For $\alpha = 18$ or 26, define $P(\alpha,S)$ to be the subcomplex of $G(S)$ such that $|P(\alpha,S)| = C^*(\alpha,6,S)$. If $R$ is a complex and $R_1$ is the 1-skeleton of $R$ then we say $R$ is $\alpha$-full with respect to $S$ if $P(\alpha,S) \subseteq R \subseteq G(S)$ and $R = G_2(R_1)$.

Let $K$ be any unit cell and let $O$ be any complex that is 26-full with respect to $SNK$. It is readily confirmed (by inspecting the 22 types of unit cell) that $C^*(26,6,SNK)$ is a shrunken image of $|O|$, and $|O|\setminus C^*(26,6,SNK) \subseteq \text{int}(K)$. (In fact this assertion is vacuous unless $(K,S)$ is of type 2, 3, 4, 6 or 9, for in all other cases $|P(26,SNK)| = C^*(26,6,SNK) = |G(SNK)|$.) Similarly, if $(K,S)$ is not of type 5, and $O$ is 18-full with respect to $SNK$, then $C^*(18,6,SNK)$ is a shrunken image of $|O|$, and $|O|\setminus C^*(18,6,SNK) \subseteq \text{int}(K)$.

But it is easy to see that if $p \in SNK$ (where $K$ is any unit cell) and $J(K,S)$ is the subcomplex of $P(\alpha,SNK)$ consisting of all the simplexes of $P(\alpha,SNK)$ that do not have $p$ as a vertex, then $J(K,S)$ is $\alpha$-full with respect to $SNK\setminus\{p\}$. Hence $C^*(\alpha,6,SNK\setminus\{p\})$ is a shrunken image of $|J(K,S)|$, and $|J(K,S)|\setminus C^*(\alpha,6,SNK\setminus\{p\}) \subseteq \text{int}(K)$.(*) (This is so even if $\alpha = 18$ and $(K,S\setminus\{p\})$ is of type 5, for then $(K,S)$ is of type $\gamma$, and we see instantly that $|J(K,S)| = C^*(18,6,SNK\setminus\{p\}) = KNS\setminus\{p\}$.)

Let $L = P(\alpha,S) = \bigcup\{P(\alpha,SNK) | K \text{ is a unit cell}\}$, and let $J = \bigcup\{J(K,S) | K \text{ is a unit cell}\}$. Then $|L| = C^*(\alpha,6,S)$ and $J$ is the subcomplex of $L$ consisting of all the simplexes in $L$ that do not have $p$ as a vertex. By (*), and the above lemma, $C^*(\alpha,6,S\setminus\{p\})$ is a shrunken image of $|J|$. Therefore $C^*(\alpha,6,S)$ is $(p,S)$-consistent.

**Proof of (iii) and (iv)**

Let $\beta$ be equal to 18 or 26.

Suppose $p \in \mathcal{Z}_k^3$. Define a total order $\ll$ on $\mathcal{Z}_k^3$ such that $p$ is the $\ll$-least point in $\mathcal{Z}_k^3$. 

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We shall define a complex \( L(V) \) for every set \( V \) of lattice points, such that, whenever \( V \) is contained in a unit cell \( K \), \( L(V) \) is a regular dissection of \( C^0(S, \beta, V) \); we shall then prove that \( L = L(V) \) has the property stated in the above remark. \( L(V) \) is defined as follows:

(a) \( L(\emptyset) = \emptyset \)

In (b) to (f) let \( q = q(V) \) be the \( \mathbb{R} \) - least point in \( V \). Then \( L(V) \) is the minimal complex such that:

(b) \( L(V) \supset L(V\setminus q) \)

(c) Each straight line segment joining \( q \) to a \( 6 \)-neighbour of \( q \) in \( V\setminus q \) is a 1-simplex of \( L(V) \)

(d) If \( F \) is one of the 12 faces of unit cells such that \( q \in F \), and every corner of \( F \) is in \( V \), let \( D \) denote the diagonal of \( F \) that contains \( q \); then the two \((1, 1, 2)\) triangles in \( F \) whose common hypotenuse is \( D \) are \( 2 \)-simplexes of \( L(V) \)

(e) If \( 0 = 18 \), and \( K \) is any unit cell in \( N(q) \) such that \((K, V) \) of type 18, 21 or 22, then all the \((\sqrt{3})/2, (\sqrt{3})/2, 1) \) triangles whose corners are the centroid of \( K \) and two mutually \( 6 \)-adjacent points in \( V\setminus K \) are \( 2 \)-simplexes of \( L(V) \)

(f) If \( K \) is any unit cell in \( N(q) \), and \((K, V) \) is of type 21 or 22, then each tetrahedron with a vertex at the centroid of \( K \) whose base with respect to that vertex is either one of the \((1, 1, 2) \) triangles contained in \( K \) that are \( 2 \)-simplexes of \( L(V\setminus q) \), or one of the \((1, 1, 2) \) triangles in \( K \) mentioned in rule (d), is a 3-simplex of \( L(V) \)

For all sets of lattice points \( S \) let \( C(S) \) denote \( C^0(S, \beta, S) \), and for all \( q \) in \( S \) let \( J(S,q) \) denote the complex \( (L(S) \setminus N_{L(q)}(q)) \cup k_{L(q)}(q) \). Suppose \( p \in U \subseteq Z_q^3 \), and in the case \( \beta = 25 \) suppose further that, for each unit cell \( K \) in \( N(p) \), either all the corners of \( K \) are in \( U \) or \( p \) is \( 18 \)-adjacent to \( U\setminus K \). It is readily confirmed that for all unit cells \( K \) in \( N(p) \) \( |L(U\setminus K(p))| \) is a shrunk image of \( |J(U\setminus K(p))| \), and \( |J(U\setminus K)| \setminus |L(U\setminus K(p))| \subseteq \text{int}(K) \). By the above lemma, this implies that \( C(U\setminus K(p)) = U(C(U\setminus K(p) | K \text{ is a unit cell}) = U\setminus L(U\setminus K(p)) | K \text{ is a unit cell}) \) is a shrunk image of \( U\setminus J(U\setminus K, p) | K \text{ is a unit cell} \)

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cell} = |J(U,p)|. 

The following is a fundamental result on sequential shrinking:

**THEOREM 4**

Let \( C(A(.),.) \) be a family of polyhedra in \( R^n \), let \( S \) be a set of lattice points in \( Z^n \) and let \( p \in S \) be a point on the frontier of \( C(A(S),S) \) such that \( C(A(.),.) \) is \((p,S)\)-consistent. Let \( L \) be a complex such that \(|L| = C(A(S),S)\) and such that if \( J \) is the complex \((L \setminus L_k(p)) \cup L_k(p)\) then \( C(A(S \setminus \{p\}),S \setminus \{p\}) \) is a shrunken image of \(|J|\). (Existence of such an \( L \) is equivalent to \((p,S)\)-consistency of \( C(A(.),.) \), as we remarked earlier.)

Then:

(i) If \( d = 2 \), \( (A(S \setminus \{p\}),S \setminus \{p\}) \) is a shrunken image of \((A(S),S)\) if and only if \(|L_k(p)|\) is connected

(ii) If \( d = 3 \), \( (A(S \setminus \{p\}),S \setminus \{p\}) \) is a shrunken image of \((A(S),S)\) if and only if \(|L_k(p)|\) is connected and \(|\chi(|L_k(p)|)| = 1\)

**PROOF**

We shall prove (ii); (i) can be proved in the same way, but is easier.

"If"

Suppose \(|L_k(p)|\) is connected and \(|\chi(|L_k(p)|)| = 1\). We assert that if \( H \) is any subcomplex of \( L_k(p) \) that contains a 2-simplex, then \( H \) has a "free" 2-simplex -- that is, a 2-simplex one of whose edges is an edge of no other 2-simplex in \( H \). The truth of this assertion should be fairly obvious by geometric intuition; a rigorous justification could be given along the following lines:

Let \( Y \) be a (spherical) ball with centre \( p \) such that \(|L_k(p)| \subseteq \text{int}(Y)\). For all points \( x \) in \( Y \) let \( \Pi(x) \) denote the projection of \( x \) from \( p \) or the boundary of \( Y \) -- that is, the point on \( \text{Bd} Y \) such that the straight line segment (radius) joining \( p \) to \( \Pi(x) \) passes through \( x \). Then by the definition of \( L_k(p) \), \( \Pi \) is 1-1 on \( L_k(p) \). Moreover, since \( p \) is on
the boundary of $C(S, S)$, $L_k(p)$ does not surround $p$, and so $π(π(L_k(p)))$ is a proper subset of $Bd Y$. It follows that, if $W$ is the union of all the 2-simplexes in $H$, then there exists a point $z$ on the relative boundary of $π(W)$ in $Bd Y$. It is easy to see that $z$ must lie on a "free 2-simplex of $π(H)$" (the quotes are there because $Bd Y$ is a sphere and so $π(H)$ is a 'spherical simplicial complex'). There must exist a point $v$ in $W$ such that $π(v) = z$, and since $π$ is 1-1 on $W$, $v$ lies on at least one free 2-simplex of $H$. This justifies our assertion.

Define $H_0 = L_k(p)$. For all $i$ such that $H_i$ contains a 2-simplex let $H_{i+1}$ be $H_i \setminus \{σ^2_i, v^1_i\}$, where $(σ^2_i, v^1_i)$ is a two-dimensional free pair of simplexes in $H_i$. (If there is more than one free pair of simplexes then an arbitrary choice is made.) Since $L_k(p)$ contains only finitely many 2-simplexes, there exists an integer $n$ such that $H_n$ contains no 2-simplexes.

By construction, $|H_n|$ is a shrunken image of $|L_k(p)|$. Hence, by Theorem 1, $|H_n|$ connected and $X(|H_n|) = 1$. Therefore $H_n$ is a tree. For all $i ≥ n$ such that $H_i$ contains a 1-simplex (edge) pick a vertex $v_i$ of degree 1 in $H_i$, and let $e_i$ be the unique edge of $H_i$ incident on $v_i$; then let $H_{i+1}$ be the tree $H_i \setminus \{e_i, v_i\}$. Since $H_0$ contains only finitely many vertices there is an integer $m$ such that $H_m$ contains just one point, $q$ say.

By definition of $L_k(p)$, we know that for every $0 < i < n$, $σ^2_i$ is a two-dimensional face of a 2-simplex $τ^2_i$ in $L$ which has a vertex at $p$, and $σ^1_i$ is an edge of a 2-simplex $τ^2_i$ in $L$ which has a vertex at $p$. Similarly, for every $n < i < m$, $e_i$ is an edge of a 2-simplex $τ^2_i$ in $L$ with a vertex at $p$, and $v_i$ is an endpoint of a 1-simplex $τ^1_i$ in $L$ whose other endpoint is $p$. Finally, $q$ is an endpoint of a 1-simplex $τ^1_q$ in $L$ whose other endpoint is $p$.

Define $L_0 = L$, and for $0 < i < n$ define $L_{i+1} = L_i \setminus \{τ^2_i, τ^2_{i+1}\}$; for each $n < i < m$ define $L_{i+1} = L_i \setminus \{τ^2_i, τ^1_i\}$. Finally, define $L_{m+1} = L_m \setminus \{σ^2, τ^1_q\}$. It is easily verified that, for all $0 < i < m$, $L_i$ is a subcomplex of $L$, and the link of $p$ in $L_i$ is $H_i$. In particular, the link of $p$
in \( L \) is \( \{q\} \). Plainly, every simplex of \( L \) that does not contain \( p \) is a simplex of \( L_i \) for all \( 0 \leq i \leq m \), and, in particular, of \( L_m \). Hence the subcomplex \( J \) of \( L \) defined in the statement of the theorem is just \( L_m \).

It is readily confirmed that for all \( 0 \leq i \leq m \), \( |L_{i-1}| = \text{COLLAPSE}(|L_i|, \tau_0^i, \tau_1^i) \), and for every \( n \leq i \leq m \), \( |L_{i-1}| = \text{COLLAPSE}(|L_n|, \tau_0^i, \tau_1^i) \). Finally, we have \( |L_m| = \text{COLLAPSE}(|L_n|, \tau_0^m, \{q\}) \). Hence \( |J| = |L_m| \) is a shrunken image of \( |L| = \text{C}(A(S), S) \), and so \( \text{C}(A(S\setminus\{p\}), S\setminus\{p\}) \) is a shrunken image of \( \text{C}(A(S), S) \).

**Only if**

Suppose \( \text{C}(A(S\setminus\{p\}), S\setminus\{p\}) \) is a shrunken image of \( \text{C}(A(S), S) \). Then \( |J| = |L| \cup |N(p)| \), and \( \text{JAN}_N(p) = \text{Lk}_L(p) \). Moreover, since \( p \) is a shrunken image of \( |N(p)| \), \( X(|N(p)|) = 1 \). Hence \( X(|J|) = X(|J|) + 1 - X(|\text{Lk}_L(p)|) \). But \( \text{C}(A(S\setminus\{p\}), S\setminus\{p\}) \) is a shrunken image both of \( |J| \) and of \( |L| = \text{C}(A(S), S) \), whence \( X(|J|) = X(|L|) \). It follows that \( X(|\text{Lk}_L(p)|) = 1 \).

It remains to show that \( |\text{Lk}_L(p)| \) is connected. Since \( \text{C}(A(S\setminus\{p\}), S\setminus\{p\}) \) is a shrunken image of \( \text{C}(A(S), S) \) there exists a sequence of polyhedra \( P_0 \supset P_1 \supset \cdots \supset P_n \) (\( n \geq 1 \)), where \( P_0 = \text{C}(A(S), S) \) and \( P_n = \text{C}(A(S\setminus\{p\}), S\setminus\{p\}) \), with the property that for all \( 0 \leq i < n \) there exist simplices \( C_i \) and \( F_i \) such that \( P_{i+1} = (P_i, C_i, F_i) \). For each \( 0 \leq i < n \) let \( M_i \) be a complex and let \( (\sigma_i, \tau_i) \) be a free pair of simplices of \( M_i \) such that \( |M_i| = P_i \) and \( |M_i \setminus (\sigma_i, \tau_i)| = P_{i+1} \). Let \( M_n = M_{n-1} \setminus (\sigma_i, \tau_i) \), so that \( |M_n| = P_n \). Let \( W_0 \) be a complex such that \( \{p\} \) is a 0-simplex of \( W_0 \), and such that for all \( i \) there exists a subcomplex \( W_i \) of \( W_0 \) such that \( W_i \) is a subdivision of \( M_i \). Plainly \( \{p\} \) is not a 0-simplex of \( W_0 \). Let \( m \) be the largest integer such that \( \{p\} \) is a 0-simplex of \( W_m \). For brevity, let \( |\text{Lk}_L(p)| \) denote the link of \( p \) in \( W_m \). Consider the following assertion:

There exist 0-simplices \( \{x_i\} \) and \( \{y_i\} \) of \( \text{Lk}_L(p) \) such that \( x_i \) and \( y_i \) are contained in different components of the set \( |\text{Lk}_L(p)| \), but are contained in the same component of \( P_i \setminus \{p\} \). (***)

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It is easy to see that (*) is false in the case $i = m$. For, by the definition of $m$, $p$ lies in $σ_*$ and is not in the closure of $P_n \setminus σ_*$, from which it follows that $Lk_*(p)$ is connected. Thus to show that $|Lk_*(p)|$ is connected, it suffices to deduce the validity of (*) for $|J|$ from the assumption that $|Lk_*(p)|$ is not connected.

We begin by showing that if $|Lk_*(p)|$ is not connected then (*) must hold when $i = 0$.

Let $O$ denote the component of $P_w \setminus L(A(S), S)$ to which $p$ belongs. Since $P_n$ is a shrunken image of $P_0$, Theorem 1 implies that $O$ contains just one component of $P_n$. But $P_n = C(A(S \setminus p), S \setminus p))$ is also a shrunken image of $|J|$, so that each component of $|J|$ contains exactly one component of $P_n$. Therefore $O$ contains just one component of $|J|$; thus $O \cap |J|$ is a component of $|J|$. Let $S(p, ε)$ denote the (surface of the) sphere with centre $p$ and radius $ε$. By hypothesis $|Lk_*(p)|$ is not connected, so for all sufficiently small $ε$ the set $|L| \cap S(p, ε) = |L| \cap S(p, ε)$ is not connected. Therefore $|Lk_*(p)|$ is not connected. Let $<x_1>$ and $<y_1>$ be 0-simplexes of $Lk_*(p)$ such that $x_1$ and $y_1$ are contained in different components of the set $|Lk_*(p)|$. Since $|Lk_*(p)| \subset O \cap |J|$, and $O \cap |J|$ is a component of $|J|$, $x_1$ and $y_1$ are contained in the same component of $|J|$, and hence of $P_w \setminus \{p\}$. This shows that (*) holds when $i = 0$.

Suppose (as an induction hypothesis) that (*) holds for $i = k$, where $k < m$. Consider the case $i = k+1$. Let $<x_k>$ and $<y_k>$ be 0-simplexes of $Lk_*(p)$ such that $x_k$ and $y_k$ are contained in different components of the set $|Lk_*(p)|$, but are contained in the same component of $P_w \setminus \{p\}$. Then there is a path $Γ_k$ in the 1-skeleton of $W_k$ from $x_k$ to $y_k$ that does not pass through $p$. If $Γ_k$ does not meet $σ_k$, then $Γ_k$ is also a path in the 1-skeleton of $W_{k+1}$, so (*) is valid for $i = k+1$ (because we can define $x_{k+1} = x_k$ and $y_{k+1} = y_k$). Notice that this covers the case in which $(σ_k, τ_k)$ is a 1-dimensional free pair of simplexes in $M_k$, for then $Γ_k$ cannot meet $σ_k$.

Now consider the case in which $(σ_k, τ_k)$ is a 2- or 3-dimensional free pair of simplexes in $M_k$ and $Γ_k$ meets the simplex $σ_k$. Let $B_k$ denote the set of points on the relative
boundary of \( \sigma \) that is not in the relative interior of \( \tau \). Thus \( B_k = \sigma \cap P_k \). There are two sub-cases: either \( p \) lies in \( \sigma \), or \( p \) does not lie in \( \sigma \).

Suppose \( p \) lies in \( \sigma \). Then \( p \) lies in \( B_k \) since \( k < m \) implies \( p \in P_k \). Hence \( \sigma \) meets just one component of \( \text{lk}_k(p) \). Since \( p \) and \( q \) belong to different components of \( \text{lk}_k(p) \), we may assume without loss of generality that \( p \) does not belong to the component of \( \text{lk}_k(p) \) that meets \( \sigma \). This implies (a fortiori) that \( p \neq q \). Thus \( x \in P_k \) and \( \{x_1\} \) is a 0-simplex of \( \text{lk}_k(p) \). Let \( u \) be the first vertex of \( \Gamma \) (counting from the \( x \) end of \( \Gamma \)) that lies on the relative boundary of \( \sigma \). Since \( (\sigma, \tau) \) is a free pair of \( M_k \), \( u \in B_k \). There exists at least one 1-simplex of \( \text{W}_k \) which is contained in \( B_k \), which has an endpoint at \( p \), and whose other endpoint, which we shall call \( q \), lies in the same component of \( B_k \backslash \{p\} \) as \( u \). Plainly \( q \in P_k \), and \( q \) lies in the same component of \( P_k \backslash \{p\} \) as \( x \). On putting \( x_{k+1} = x \) and \( y_{k+1} = q \), we see that \((*)\) is valid for \( i = k+1 \).

Suppose instead that \( p \) does not lie in \( \sigma \). If either \( x \) or \( y \) lies in the relative interior of \( \sigma \) or \( \tau \), then -- since \( p \notin \sigma \) and \( \{x_1\}, \{y_1\} \in \text{lk}_k(p) \) -- \((\sigma, \tau)\) is not a free pair of simplexes in \( M_k \), a contradiction. So neither \( x \) nor \( y \) can lie in the relative interior of \( \sigma \) or \( \tau \). Thus \( x \) and \( y \) are both contained in \( P_k \), and they are both 0-simplexes of \( \text{lk}_k(p) \). Let \( s \) and \( t \) be respectively the first and last vertices of \( \Gamma \) (counting from the \( x \) end) that lie in \( \sigma \). Since \( (\sigma, \tau) \) is a free pair in \( M_k \), \( s \) and \( t \) are both on \( B_k \).

Since (by hypothesis) \( p \notin B_k \), \( s \) and \( t \) (and hence \( x \) and \( y \)) belong to the same component of \( P_k \backslash \{p\} \). On putting \( x_{k+1} = x \) and \( y_{k+1} = y \), we see that \((*)\) holds for \( i = k+1 \).

We conclude that if \( |\text{lk}_k(p)| \) is not connected then \((*)\) must hold for all \( 0 \leq i \leq m \). As we remarked above, this is false, and so \( |\text{lk}_k(p)| \) is connected. \( \square \)
COROLLARY

Let \( C(A(.), .) \) be a family of polyhedra in \( \mathbb{R}^d \), and let \( S \) be a set of lattice points such that \( C(A(S), S) \) is a 1-, 2-, or 3-dimensional manifold with boundary in a neighbourhood of a point \( p \) in \( S \), with \( p \) on the manifold-boundary. Then if \( C(A(.), .) \) is \((p, S)\)-consistent \( C(A(S\{p\}), S\{p\}) \) is a shrunken image of \( C(A(S), S) \).

PROOF

It is well known (see the proof of Proposition 5.4.3 in [Maundy70]) that if \( L \) is any complex such that \( |L| \) is an \( n \)-manifold with boundary in a neighbourhood of a point \( x \) in \( |L| \), and \( x \) is on the manifold-boundary, then \( |L_{k}(x)| \) is homeomorphic to the closed \((n-1)\)-dimensional hemisphere. In particular, \( |L_{k}(x)| \) is connected and its Euler characteristic is \( 1 \). Hence the Corollary follows from Theorem 4. \( \square \)

The following proposition is relatively easy; but in view of Theorem 1, it is clear that it expresses an important property of simple points.

PROPOSITION 5

Let \( (A(.), .) \) be a consistent and \( X \)-regular family of digital pictures such that 
\( C^{*}(A(.), .) \) is \((p, S)\)-consistent, where \( p \) is a simple point of \( (A(.), S) \). Let \( L \) be a complex such that \( |L| = C^{*}(A(S), S) \), and such that if \( J \) is the complex \( (L\setminus N_{k}(p))\cup L_{k}(p) \) then \( C^{*}(A(S\{p\}), S\{p\}) \) is a shrunken image of \( |J| \). The existence of such a complex \( L \) is equivalent to the \((p, S)\)-consistency of \( C(A(.), .) \). Then \( X(|L_{k}(p)|) = 1 \).

PROOF

For all \( i \in \mathbb{Z}^3 \) let \( C(T) \) denote \( C^{*}(A(T), T) \). Now \( |J\setminus C(S\{p\})| \subseteq \text{int}(N(p)) \), since 
\( C(S\setminus C(S\{p\})) \subseteq \text{int}(N(p)) \), so \( C(S\{p\})\setminus N(p) \) is a shrunken image of \( |J\setminus N(p)| \). Therefore

\[
X(|J\cap N(p)|) = X(C(S\setminus N(p)))
\]

by Theorem 1

\[
= X(C(S)\setminus N(p))
\]

by condition (iii) of the definition of a simple point
\[ X((\mathcal{J}\cap N(p)) \cup N_L(p)) \]
\[ = X((\mathcal{J}\cap N(p)) + X((N_L(p))) - X((\mathcal{J}\cap L_L(p))) \]
\[ = X((\mathcal{J}\cap N(p)) + X((p)) - X((\mathcal{J}\cap L_L(p))) \]
\[ \qquad \text{because } \langle p \rangle \text{ is a shrunken image of } N_L(p) \]
\[ = X((\mathcal{J}\cap N(p)) + 1 - X((\mathcal{J}\cap L_L(p))) \].

Hence \( X((\mathcal{J}\cap L_L(p))) = 1. \]

**Proofs of the Main Theorems**

**Lemma 6**

Suppose \( \delta = 18 \) or 26, \( p \in S \subseteq \mathbb{Z}^3 \) and \( p \) is on the frontier of at most one component of \( N(p) \setminus \mathcal{C}^*(6, \delta, S) \). Then, for all \( r \leq (\sqrt{3})/2 \), \( B(p, r) \setminus \mathcal{C}^*(6, \delta, S) \) is connected, where \( B(p, r) \) denotes the open ball of radius \( r \) with centre \( p \).

**Proof**

Suppose, for the purpose of deriving a contradiction, that \( r \leq (\sqrt{3})/2 \) and the points \( x \) and \( y \) belong to different components of \( B(p, r) \setminus \mathcal{C}^*(6, \delta, S) \). Then it is easy to see that the straight line segments \( xp \) and \( yp \) meet \( \mathcal{C}^*(6, \delta, S) \) only at \( p \). (This need not be true if \( r > (\sqrt{3})/2 \) -- e.g. one of the unit cells in \( N(p) \) might be a cell of type 21 in which all corners that are 18-adjacent to \( p \) belong to \( S \).) So since \( p \) is on the frontier of at most one component of \( N(p) \setminus \mathcal{C}^*(6, \delta, S) \), there must exist a simple polygonal arc \( \gamma \) in \( N(p) \setminus \mathcal{C}^*(6, \delta, S) \) which joins \( x \) to \( y \). Let \( \langle z_i, 0 \leq i \leq n \rangle \) be a sequence of points on \( \gamma \) such that:

(i) If \( d_i \) denotes the distance from \( z_i \) to \( x \), measured along \( \gamma \), then \( \langle d_i, 0 \leq i \leq n \rangle \)

is a strictly increasing sequence.

(ii) For all \( 0 \leq i \leq n \) there exists a unit cell \( K_i \) in \( N(p) \) that contains both \( z_i \) and \( z_{i+1} \).

(iii) \( z_0 = x \) and \( z_n = y \)
Let \( U_i \) denote the component of \( K \setminus C^*(G, B, S) \) that contains \( z_i \).

One sees by inspection that, for all unit cells \( K \) in \( N(p) \), if \( O \) is any component of \( K \setminus C^*(G, B, S) \) then either \( O \) meets \( B(p, r) \), or \( O \) is a component of \( N(p) \setminus C^*(G, B, S) \). This implies that \( O \) meets \( B(p, r) \) (because the component of \( N(p) \setminus C^*(G, B, S) \) that contains \( O \), also contains \( x \) and \( y \), and therefore meets \( B(p, r) \)). One sees by inspection that, for all unit cells \( K \) in \( N(p) \), if \( R \) is any component of \( K \setminus C^*(G, B, S) \) such that \( R \) meets \( B(p, r) \) then \( R \cap B(p, r) \) is connected. On putting \( K = K_i \) and \( R = O \), we deduce that \( O \cap B(p, r) \) is connected.

If \( K_i = K_{i+1} \), then \( O_i = O_{i+1} \); otherwise \( O_i \cap O_{i+1} \) is a non-empty \((z_i, z_{i+1})\) union of components of \((K_i \setminus K_{i+1}) \setminus C^*(G, B, S)\). Since \( z_{i+1} \in K_i \setminus K_{i+1} \), \((K_i \setminus K_{i+1}) \setminus C^*(G, B, S)\) is non-empty.

But \((K_i \setminus K_{i+1}) \setminus C^*(G, B, S)\) is non-empty implies that both \((K_i \setminus K_{i+1}) \setminus C^*(G, B, S)\) and \((K_i \setminus K_{i+1}) \setminus C^*(G, B, S) \cap B(p, r)\) are non-empty and connected (this is a special property of \( C^*(G, B, S) \), as is easily seen). Therefore either \( O_i = O_{i+1} \) or \( O_i \cap O_{i+1} \cap B(p, r) \cap B(p, r) \) is non-empty and connected. Since we have already seen that \( O_i \cap B(p, r) \cap B(p, r) \) is connected, we deduce that

\[ \bigcup_{0 < i < n} O_i \cap B(p, r) \cap B(p, r) \] is connected. But \( x \) and \( y \) both belong to

\[ \bigcup_{0 < i < n} O_i \cap B(p, r) \cap B(p, r) \], since \( z_0 = x \) and \( z_n = y \). This contradicts the definition of \( x \) and \( y \), and so the theorem is proved. \( \blacksquare \)

**Theorem 7**

Suppose \( G = 10 \) or \( 20 \), and suppose \( p \in S \in \mathbb{Z}_n^2 \). Suppose further that \( p \) satisfies conditions (ii) and (iii) in the definition of an \((A(\cdot), S)\) simple point with \((A(\cdot), S) = (G, B, S)\). Then \( p \) is a simple point of \((G, B, S)\) and \((G, B, S \setminus \{p\})\) is a shrunk image of \((G, B, S)\).
PROOF

Suppose \( B = \mathbb{Z} \). If \( S^5 \cap N(p) \) contains just one point, and this point is not an 10 neighbour of \( p \), then we see at once that Theorem 7 is true. So we may ignore this case in the rest of the argument.

Now if \( B = \mathbb{Z} \) then condition (iii) of the definition of a simple point implies that, for each unit cell \( K \) in \( N(p) \), either every corner of \( K \) is in \( S \), or \( p \) is 18-adjacent to \( S^5 \cap K \). Hence \( C^*(6, \mathbb{Z}_6, \ldots) \) is \((p,S)\)-consistent by Proposition 3(iv). If on the other hand \( B = 18 \) then \( C^*(6, \mathbb{Z}_6, \ldots) \) is \((p,S)\)-consistent by Proposition 3(iii).

For all \( T \in \mathbb{Z}_q \) let \( C(T) \) denote \( C^*(6, 8, T) \). Since \( C^*(6, 8, \ldots) \) is \((p,S)\)-consistent, there exists a complex \( L \) such that \( |L| = C(S) \), and such that if \( J \) is the complex \( (L \cap N(p)) \cup L_k(p) \) then \( C(S \setminus \{p\}) \) is a shrunken image of \( |J| \).

Our aim is to deduce Theorem 6 from Theorem 4. By Proposition 5, \( X(|L_k(p)|) = 1 \). So let us suppose, for the purpose of deriving a contradiction, that \(|L_k(p)|\) is not connected.

Since the Euler characteristic of a polyhedron is the sum of the Euler characteristics of its components, and since \( X(|L_k(p)|) = 1 \), there must be a component of \(|L_k(p)|\) whose Euler characteristic is less than 1. Let \( M \) be a subcomplex of \( L_k(p) \) such that \(|M|\) is a component of \(|L_k(p)|\) and \( X(|M|) < 1 \). Let \( H \) be the subcomplex of \( N(p) \) consisting of \( \{p\} \) and all the simplices of \( N(p) \) that contain a simplex of \( M \). (Thus \( L_k(p) = M \).

For all positive \( \varepsilon \) let \( S(p, \varepsilon) \) denote the 2-sphere of radius \( \varepsilon \) with centre at \( p \). Then, for all sufficiently small \( \varepsilon \) \(|H| \cap S(p, \varepsilon) \) is homeomorphic to \(|M|\). Hence \( X(|H| \cap S(p, \varepsilon)|) < 1 \). It follows that for small \( \varepsilon \) \( S(p, \varepsilon) \setminus |H| \) is not connected. But \(|H| \cap B(p, \varepsilon) \) is the union of all straight line segments joining \( p \) to a point in \(|H| \cap S(p, \varepsilon) \). (Here \( E(p, \varepsilon) \) is the open ball of radius \( \varepsilon \) and centre \( p \).) Hence \( B(p, \varepsilon) \setminus |H| \) is not connected for all
sufficiently small $\varepsilon$. This implies that $B(p,\varepsilon) \setminus \{L\}$ is not connected for all
sufficiently small $\varepsilon$. So by Lemma 6 $p$ is on the frontier of two different components $C_1$
and $C_2$ of $N(p) \setminus \{L\}$.

By the definition of a continuous analogue, each of $C_1$ and $C_2$ contains a lattice point. In
the case $B = 26$, these lattice points are $B$-adjacent to $p$, which violates condition (ii) of the
definition of a simple point, since by the definition of a continuous analogue $C_1 \cap N_0^B$ and $C_2 \cap N_0^B$ are $B$-components of $S^0N(p)$.

In the case $B = 18$, we claim that each of $C_1$ and $C_2$ contains a lattice point that is
18-adjacent to $p$. For suppose $q \in C_1 \cap N_0^B$ (where $j = 1$ or 2) and $q$ is not 18-adjacent to
$p$. Then at least one of the 6-neighbours of $q$ in $N(p)$ must be in $S^0$, for otherwise the
unit cell containing $p$ and $q$ would be of type 21, and we see from the definition of
$C^*(6,18,S)$ that $p$ would not lie on the frontier of the component of $N(p) \setminus C^*(6,18,S)$ that
contained $q$, contrary to the definition of $q$. Since each 6-neighbour of $q$ in $S^0N(p)$ is
18-adjacent to $p$, and belongs to the same component of $N(p) \setminus C^*(6,18,S)$ as $q$, our
claim is justified. This again violates condition (ii) of the definition of a simple point.

These contradictions prove that $|Lk_+(p)|$ is connected, and so we are home by Theorem 4
and Proposition 2. 

**COROLLARY 1**

If $B = 18$ or 26 then $(6,8,S \setminus \{p\})$ is a shrunk image of $(6,8,S)$ if and only if $p$
satisfies conditions (ii) and (iii) in the definition a simple point of $(6,8,S)$.

**PROOF**

This is an immediate consequence of Theorem 7 and Proposition 2. 

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COROLLARY 2

A point is deletable in $S$ in the sense of [TourlakisMylopoulos73] if and only if it is a simple point of $(6,26,S)$.

PROOF (sketch)

By the "(a)" is equivalent to (c)" part of Proposition 3.8 in [TourlakisMylopoulos73] and Theorem 4 in this chapter $p$ is deletable in the sense of Tourlakis and Mylopoulos if and only if $(6,26,S\backslash(p))$ is a shrunk image of $(6,26,S)$. Corollary 2 now follows from Corollary 1. ☐

LEMMA 8

Suppose $a = 18$ or $26$, and suppose $p \in S \subseteq Z_n^3$. Let $X$ be the component of $C^*(a,6,S) \cap N(p)$ that contains $p$. Then $X(x) = 1$.

PROOF

The inclusion-exclusion principle implies that

$$X(x) = \sum \{X(xD) \mid D \text{ is a unit cell containing } p\}$$

$$- \sum \{X(xD) \mid D \text{ is a face of a unit cell and } p \notin D\}$$

$$+ \sum \{X(xD) \mid D \text{ is an edge of a unit cell and } p \notin D\}$$

$$- X(p)$$

But if $D$ is a unit cell, or an edge or a face of a unit cell, and $p \in D$, then it is easily verified that $X(xD) = 1$. Hence (*) shows that $X(x) = 8 - 12 + 6 - 1 = 1$. ☐

THEOREM 9

Suppose $a = 18$ or $26$, and suppose $p \in S \subseteq Z_n^3$. Suppose further that $p$ satisfies conditions (i) and (iii) of the definition of an $(A(\cdot),S)$ simple point with $(A(\cdot),S) = (a,6,\cdot)$. Then $p$ is a simple point of $(a,6,S)$ and $(a,6,S\backslash(p))$ is a shrunk image of $(a,6,S)$.
PROOF

For all $T \in \mathbb{N}$, let $C(T)$ denote $C^t(\alpha, G, T)$. Since $C^t(\alpha, G, T)$ is $(p, S)$-consistent (by Proposition 3), there exists a complex $L$ such that $|L| = C(S)$, and such that if $J$ is the complex $(L \setminus N_L(p)) \cup N_L(p)$ then $C(S \setminus \langle p \rangle)$ is a shrunken image of $|J|$. By Proposition 5, $X(|L_L(p)| = 1$. In view of Theorem 4, it only remains to prove that $|L_L(p)|$ is connected.

Suppose otherwise. Then since $X(|L_L(p)| = 1$ there exists a subcomplex $M$ of $L_L(p)$ such that $|M|$ is a component of $|L_L(p)|$ and $X(|M|) \geq 1$. Let $H$ be the subcomplex of $N_L(p)$ consisting of $\langle p \rangle$ and all the simplices of $N_L(p)$ that contain a simplex of $M$.
(Thus $L_L(p) = M$.). Let $G$ be the complex $(N_L(p) \setminus \{\langle p \rangle\})$. (Thus $L_L(p) = |L_L(p)| \setminus \{\langle p \rangle\}.$)

Let $X$ denote the component of $C(S \setminus N(p))$ that contains $p$. Now $(C(S \setminus N(p))^C$ is connected, since each component of $N(p) \setminus C(S)$ contains a lattice point. Hence $X^C$ is connected (for $\alpha = 2\alpha$, then $C(S \setminus N(p) = X$, and if $\alpha = 1$ then $C(S \setminus N(p) = X \cup R$ where $R$ is a set of isolated points). It follows from this that $(X^C\cup \{\langle p \rangle\})^C = X^C \cup H$ is connected. In other words $X^C(\{\langle p \rangle\})$ has no cavities.

Now $C(S \setminus \{p\})$ is a shrunken image of $|J|$, and $|J| \setminus C(S \setminus \{p\}) \not\subseteq C(S) \setminus C(S \setminus \{p\}) = \text{int}(N(p))$. Hence (by Theorem 1) $|J| \setminus N(p)$ has just as many components as $C(S \setminus \{p\})$. By condition (i) of the definition of a simple point, $C(S \setminus \{p\}) \setminus N(p)$ has exactly as many components as $C(S) \setminus N(p)$. So $|J| \setminus N(p)$ has just as many components as $C(S) \setminus N(p)$, which implies that $X$ contains just one component of $|J| \setminus N(p)$. Thus $X^C(J)$ is connected, whence $X^C(\{\langle p \rangle\})$ is connected, since $|G|$ is connected and meets $|J|$.

As $X^C(J \cup \{p\})$ is connected and has no cavities, $X(X^C(J \cup \{p\})) \leq 1$. Moreover, $X(|H|) = 1$ ($\{p\}$ is a shrunken image of $|H|$) and $X(|M| \cup \{p\}) = X(|M|) + 1 \geq 2$, by the definition of $M$. So since $X = (X^C(J \cup \{p\}) \cup |H|), \text{and} \ |H| \cup X^C(J \cup \{p\}) = |M| \cup \{p\}$.

$X = X(X^C(J \cup \{p\})) + X(|H|) = X(|M| \cup \{p\}) \leq 1 + 1 - 2 = 0$, contrary to
Lemma 8. □

COROLLARY

If \( \alpha = 26 \) or 18 then \((0, 5.5 \setminus \rho)\) is a shrunken image of \((0, 5.5)\) if and only if \( \rho \) satisfies conditions (i) and (iii) in the definition a simple point of \((0, 5.5)\).

PROOF

This is an immediate consequence of Theorem 9 and Proposition 2. □
Chapter 5 - SURFACE POINTS

In recent papers [MorgenthalerRosenfeld81, ReedRosenfeld82, Reed84] Morgenthaler, Reed and Rosenfeld introduced the notion of a 'simple surface point'. These authors defined surface points only in the cases where either 6-adjacency is used for object points and 26-adjacency for background points, or 26-adjacency is used for object points and 6-adjacency for background points. We generalize their concept in an obvious way to the cases in which α-adjacency is used for object points and β-adjacency for background points, where α and β are drawn from the set \{6, 1B, 2B\}. In our notation the surface points considered by Morgenthaler, Reed and Rosenfeld become \((6, 26)\)-surface points and \((26, 6)\)-surface points.

Surface points are defined axiomatically, and it is difficult to fully understand the geometric meaning of this concept just by reading the axioms. The main aim of this chapter is to prove 'structure theorems' which relate \((α, β)\)-surface points to polyhedral surfaces. Such theorems are established for eight of the nine possible combinations of \((α, β)\), the exception being the case \((α, β) = (6, 6)\). We show, for example, that if \((α, β) \neq (6, 6)\) and every \(p\) in \(S\) is an \((α, β)\)-surface point of \(S\) then there is a continuous analogue of \((α, β, S)\) which is a polyhedral surface (without boundary).

We use the structure theorems to give geometric proofs of some basic properties of surface points. In particular, Proposition 13 is a discrete "Jordan-Brouwer Separation Theorem", special cases of which were first proved in [MorgenthalerRosenfeld81, ReedRosenfeld82, Reed84] by quite difficult combinatorial arguments. We also show (in the corollary to Proposition 15) that when \((α, β) \neq (6, 6)\) one of the three axioms used by the earlier authors to define surface points can be deduced from the other two.

Counterexamples are given to confirm that results proved for \((α, β) \neq (6, 6)\) do indeed fail when \((α, β) = (6, 6)\). Finally, the structure theorems reveal the geometric meaning of the "cross-cap", which was a major obstacle in the proof of the discrete Jordan-Brouwer Separation Theorem by direct methods.

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In the cases \((a,b) = (26,6)\) and \((18,6)\) surface points may have an application to three-dimensional thinning. Consider a digital picture \((a,6,5)\) where \(a = 18\) or 26.

Let \(N(S)\) denote the points \(p\) in \(S\) such that

(i) \(p\) is \(6\)-adjacent to just one \(6\)-component of \(S^c \cap N(p)\)

and (ii) \(p\) is an \((a,6)\)-surface point of the \((6,5^c)\)-border* of \(S\).

Theorems 11 and 12 in this chapter suggest that \(N(S)\) may be a three-dimensional analogue of the set of 'non-multiple pixels' defined by Arcelli in his paper [Arcelli81] on two-dimensional thinning by contour tracing. In the case \((a,6) = (18,6)\) the comparatively fast three-dimensional border tracking algorithm due to Artzy, Frieder and Herman [Artzy,Frieder,Herman81] might be used in any three-dimensional analogue of Arcelli's algorithm.

* Recall (from Chapter 2) that the \((6,5^c)\)-border of \(S\) is the set

\[ \{ q \in S \mid q \text{ is } 6\text{-adjacent to } S^c \}. \]

Basic Concepts

Surface-Points and Digital-Surfaces

In [Morgenthaler,Rosenfeld81], [Reed,Rosenfeld82] and [Reed84], Morgenthaler, Reed and Rosenfeld define 'simple surface points' and 'simple closed surfaces'. These definitions express an extension of a three-dimensional analogue of the property of \(t\)-curves given in Proposition 3.3 of [Rosenfeld73]. The definitions can be generalized to arbitrary pairs of "pure" adjacency relations \((a,b)\) as follows (see the remarks after Propositions 14 and 15):

Let each of \(a\) and \(b\) denote one of integers \(5, 18\), or 26. If \(p \in W \subset \mathbb{Z}^2\) then we call \(p\) an \((a,b)\) surface-point of \(W\) if \(p\) is \(b\)-adjacent to exactly two \(b\)-components of \(W^c\langle p \rangle\) and each \(a\)-neighbour of \(p\) contained in \(W\) is \(b\) adjacent to both of these \(b\)-components.
We say \( W \) is an \((0, B)\) digital-surface if every point in \( W \) is an \((0, B)\) surface-point of \( W \).

The basic goal of this chapter is to present a visual interpretation of surface-points (except in the case \( c = B = 6 \)), and to use this visual interpretation to give very natural proofs of several non-trivial theorems about digital-surfaces. One of the results proved is that if \( c \) and \( B \) are not both equal to 6 then the complement of an \((0, B)\) digital-surface is not \( B \)-connected. We also exhibit a \((5, 6)\) digital-surface whose complement is \( B \)-connected, thus showing that \((B, B)\) surface-points are quite unlike the other kinds of surface-point.

**REMARK**

Observe that a \((26, B)\) surface-point of a set \( S \) is always an \((18, B)\) surface-point of \( S \) and an \((18, B)\) surface-point of \( S \) is always an \((6, B)\) surface-point of \( S \), \( \blacksquare \)

**Polyhedral Quasi-Surfaces and Plates**

We use the term polyhedral quasi-surface (with boundary) to denote a polyhedral \( [L] \) where \( L \) is a two-dimensional complex with the property that each 1-simplex in \( L \) is an edge of at least one and at most two 2-simplices in \( L \). If \( \Gamma = [L] \) is a polyhedral quasi-surface then the (surface)-boundary of \( \Gamma \) (denoted by \( \partial \Gamma \)) is the union of all the 1-simplices in \( L \) that are edges of just one 2-simplex in \( L \). (It is readily confirmed that although this definition of \( \partial \Gamma \) refers to \( L \) the set \( \partial \Gamma \) depends only on \( \Gamma \), being independent of \( L \).) We shall say that \( \Gamma \) is a polyhedral quasi-surface (without boundary) if \( \Gamma \) is a polyhedral quasi-surface such that \( \partial \Gamma = \emptyset \).

A strongly-connected polyhedral quasi-surface (with boundary) is a polyhedral quasi-surface \( \Gamma \) with the property that if \( F \) is any finite set of points then \( \Gamma \setminus F \) is
connected.

The following proposition expresses a fundamental property of polyhedral quasi-surfaces which will be used on several occasions in this chapter.

PROPOSITION 0

Let \( X \) be a closed cuboid in \( \mathbb{R}^3 \), and let \( \Gamma \) be a polyhedral quasi-surface (with boundary) contained in \( X \) such that \( \Sigma^{\text{int}}(X) \neq \emptyset \) and \( \text{Bd} \ \Sigma \subseteq \text{Bd} \ X \). Let \( p \) be any point in \( \Sigma^{\text{int}}(X) \). Then \( p \) is in the closure of two different connected components of \( X \backslash \Sigma \).

Furthermore, if \( \Sigma \) is strongly-connected and \( \text{Bd} \ \Sigma = \Sigma^{\text{int}}(\text{Bd} \ X) \) then \( X \backslash \Sigma \) has exactly two connected components, and \( \Sigma \) is a subset of the closure of each of these two sets.

REMARK

This result should be very plausible by 'geometric intuition'. A proof is outlined in the appendix to this chapter. ■

A polyhedral quasi-surface (with boundary) \( \Pi \) in \( \mathbb{R}^3 \) will be called a plate if

- either \( \Pi \) is a face of some unit cell

or \( \Pi \) is strongly-connected and satisfies each of the following three conditions:

(i) There is a unique closed unit cell in \( \mathbb{R}^3 \) which contains \( \Pi \); this cell will be denoted by \( K(\Pi) \)

(ii) \( \text{Bd} \ \Pi \) is a simple closed polygonal curve, and each straight line segment contained in \( \text{Bd} \ \Pi \) is either an edge of \( K(\Pi) \) or a diagonal of a face of \( K(\Pi) \)

(iii) \( \text{Bd} \ \Pi \subseteq \Pi^{\text{int}}(\text{Bd} \ K(\Pi)) \).

If \( \Pi \) is a face of a unit cell then we call \( \Pi \) a square plate; otherwise we call \( \Pi \) a non-square plate. If \( \Pi \) is a plate then a vertex of \( \Pi \) is any point in \( \Pi^{\text{int}} \), an edge of \( \Pi \) is a straight line segment contained in \( \text{Bd} \ \Pi \) that joins two vertices of \( \Pi \).
Plate-Cycles

If \( p \in \mathbb{Z}^2 \) then a plate-cycle at \( p \) is a sequence \( \langle \Pi_i \mid 0 \leq i \leq n \rangle \) (where \( n \geq 1 \)) of distinct plates such that:

(i) There is a sequence \( \langle e_i \mid 0 \leq i \leq n \rangle \) in which \( e_i \) and \( e_{i+1} \) are distinct edges of \( \Pi_i \) \( (0 \leq i < n) \), \( e_0 \) and \( e_n \) are distinct edges of \( \Pi_n \), and \( p \) is an endpoint of each \( e_i \).

(ii) If \( i \neq j \) then \( \Pi_i \cap \Pi_j \) is the union of a number (which may be zero) of straight line segments each of which is an edge of both plates and a set of points each of which is a vertex of both plates.

(iii) Any edge of a \( \Pi_i \) is an edge of at most one other \( \Pi_i \).

The plate-set of a plate-cycle is the set of plates in that plate-cycle. If \( P \) is an expression denoting a set of plates and \( p \in \mathbb{Z}^2 \) then \( P(p) \) denotes the set of those plates in \( P \) that have a vertex at \( p \).

Let \( p \) be a point in \( \mathbb{Z}^2 \) and let \( P \) be a set of plates such that the following are all true:

(i) If \( \Pi \in P(p) \) and \( e \) is an edge of \( \Pi \) that contains \( p \) then \( e \) is an edge of at least two plates in \( P(p) \).

(ii) If three plates in \( P(p) \) have an edge in common then \( p \) is an endpoint of one of the edges common to all three plates.

(iii) The intersection of any two distinct plates in \( P(p) \) is the union of a number (which may be zero) of straight line segments each of which is an edge of both plates and a set of points each of which is a vertex of both plates.

Then if \( \Pi' \) and \( \Pi'' \) are plates in \( P \) that have an edge in common containing \( p \) the following algorithm will generate a plate-cycle at \( p \) whose plate-set is a subset of \( P \):

**CYCLE-FINDING ALGORITHM**

Arguments \( \Pi', \Pi'' : P \)

where \( \Pi' \) and \( \Pi'' \) have an edge in common which contains \( p \).
Let $\Pi_0$ be $\Pi'$.

Let $\Pi_1$ be $\Pi''$.

Let $e_0$ be an edge common to $\Pi_0$ and $\Pi_1$ that contains $p$.

For $i=1,2,3,\ldots$.

Let $e_i$ be the edge of $\Pi_i$ that contains $p$ but is different from $e_{i-1}$; let $\Pi_{i+1}$ be a plate different from $\Pi_i$ such that $e_i$ is an edge of $\Pi_{i+1}$, until there is $m < i$ such that $e_i = e_m$.

Let $n$ be the largest $i$ for which $e_i$ is defined.

Let $m$ be the (unique) integer less than $n$ such that $e_m = e_n$.

Let result be the sequence $<\Pi_i, m<:i>n>$.  

Special Notations

(a) In a few of the proofs we use a string 'abc' (where $a$, $b$, and $c$ are digits) to denote the point with coordinates ($a$, $b$, $c$). (e.g. 201 denotes the point (2, 0, 1).) Figure 1 should make it easy to follow the proofs which use this notation.

(b) We use 'by (SP(W),2$^2$<xyz>,uvw)' [as in 'by (SP(W),2$^2$<201>,101)'] as an abbreviation for the clause 'since xyz is an (a,b) surface-point of $W$, uvw is $B$-adjacent to each of the two (and only two) $B$-components of $W$<xyz> that are $B$-adjacent to $xyz$'.

(c) If $W$ is an expression denoting a subset of $Z^3$, and $p \in Z^3$ then $W(p)$ denotes the set $W|_{p}$.

A Theory of Plates

As we mentioned in the Introduction, our approach to surface-points is based on the construction of continuous analogues. It turns out to be convenient to define a weaker kind of "continuous analogue" than we have used in previous chapters.
FIGURE 1: \( \mathbb{R}^3 \{111\} \)
DEFINITION
Let $X$ be any window, and let $(A,S)$ be any digital picture. We say that a polyhedron $C$ is a weak analogue of $(A,S)$ relative to $X$ if and only if the following conditions are satisfied for all unit cells $K$ in $X$:

(a) $S^nK \cap \text{Int}(X) \subseteq C \cap K \subseteq S^nK$

(b) If $B$ is any connected component of $K \setminus C$ then $D^n \subseteq S^nB$ is an $A$-component of $S^nK$

(c) If $D$ is a face or edge of $K$ and $D$ is not contained in $Fr X$ then each connected component of $D \setminus C$ contains a point in $S^n$

DEFINITION
Let $S$ be any subset of $Z^n$ and let $X$ be any window. A set $P$ of plates is $\beta$-natural with respect to $(S,X)$ if the following conditions both hold: (i) $UP$ is a weak analogue of $(\alpha, B, S)$ relative to $X$; (ii) for all unit cells $K$ in $X$, either $K$ does not contain a non-square plate, or $K$ contains at most one non-square plate and $K$ does not contain a square plate. We say $P$ is $\beta$-natural with respect to $S$ if $P$ is $\beta$-natural with respect to $(S, R^n)$. (Note that the value of $\alpha$ in condition (i) is in fact irrelevant.)

We observe that if $P$ is $\beta$-natural with respect to $(S,X)$ then, for all windows $Y$ such that $Y \subseteq X$, the collection of plates $\{ \pi \in P \mid \pi \subseteq Y \}$ is $\beta$-natural with respect to $(S,Y)$.

PROPOSITION 1
Let $Q$ be a window that is a closed cuboid. Let $P$ be a collection of plates which is $\beta$-natural with respect to $(S,Q)$, and let $P(Q)$ denote the set of plates in $P$ which are contained in some unit cell in $Q$. Then each component of $Q \setminus UP(Q)$ meets $S^n$ in a $\beta$-component of $S^nQ^n$.

PROOF
This only requires trivial modifications of the proof of Proposition 0(ii) in Chapter 2.
DEFINITION
Suppose \( p \in Z^2 \) and \( P \) is a collection of plates. We shall say that \( P \) satisfies the \( \beta \)-form of condition 1 with respect to \((S, p)\) iff the following sub-conditions both hold:

(a) If \( P(p) \) is the plate-set of a plate-cycle at \( p \) then \( S(p) \) has exactly two \( \beta \)-components.

(b) If \( P(p) \supseteq P \cup P' \) where \( P' \) and \( P'' \) are the plate-sets of two different plate-cycles at \( p \) then \( S(p) \) has at least three \( \beta \)-components.

The following proposition gives a sufficient condition for a set of plates to satisfy the \( \beta \)-form of condition 1.

PROPOSITION 2
Suppose \( p \in Z^2 \) and \( P \) is a collection of plates satisfying the following conditions (i), (ii), (iii):

(i) If \( \Pi \in P(p) \) then \( \Pi \cap (\text{Bd } N(p)) \) is precisely the union of the edges of \( \Pi \) that do not contain \( p \).

(ii) If \( \Pi_1, \Pi_2 \) are in \( P(p) \) then \( \Pi_1 \cap \Pi_2 \) is either \( \{p\} \) or an edge of \( \Pi_1 \) and \( \Pi_2 \) that contains \( p \) or the union of the two edges of \( \Pi_1 \) and \( \Pi_2 \) that contain \( p \).

(iii) \( P(p) \) is \( \mathcal{E} \)-natural with respect to \((S, N(p))\).

Then \( P \) satisfies the \( \beta \)-form of condition 1 with respect to \((S, p)\). Furthermore, if \( P(p) \) is the plate-set of a plate-cycle at \( p \) then \( N(p) \setminus P(p) \) has exactly two components, and if \( x \in \Pi, \epsilon P(p) \) then \( x \) is in the closure of both components.

PROOF
We shall assume throughout this proof that \( P \) satisfies (i), (ii) and (iii).
Case I

\( \mathcal{P}(p) \) is the plate-set of a plate-cycle at \( p \).

It is easy to see from the definition of 'plate-cycle' that \( \mathcal{U}\mathcal{P}(p) \) is a strongly-connected polyhedral quasi-surface (with boundary). If \( \pi \in \mathcal{P}(p) \) and \( e \) is an edge of \( \pi \) that does not contain \( p \) then by (ii) \( e \) is an edge of no other plate in \( \mathcal{P}(p) \), so \( e \in \text{Bd} (\mathcal{U}\mathcal{P}(p)) \).

Conversely, if \( e \) is an edge of a plate and \( e \in \text{Bd} (\mathcal{U}\mathcal{P}(p)) \) then by the definition of a plate-cycle \( p \not\in e \). Hence (ii) implies that \( \mathcal{U}\mathcal{P}(p) \cap (\text{Bd} \ N(p)) = \text{Bd} (\mathcal{U}\mathcal{P}(p)) \). The result now follows from Proposition 0 and Proposition 1.

Case II

\( \mathcal{P}(p) = \mathcal{P}' + \mathcal{P}'' \) where \( \mathcal{P}' \) and \( \mathcal{P}'' \) are the plate-sets of two different plate-cycles at \( p \).

By the argument given in the previous paragraph \( N(p) \setminus \mathcal{U}\mathcal{P}' \) has two connected components \( \mathcal{K}' \), \( \mathcal{K}_2 \), and \( N(p) \setminus \mathcal{U}\mathcal{P}'' \) has two connected components \( \mathcal{K}_3 \), \( \mathcal{K}_4 \). Let \( \pi \) be a plate in \( \mathcal{P}' \setminus \mathcal{P}' \) and let \( x \) be a point on \( \pi \) which is not on an edge of \( \pi \). Then \( x \) is on no other plate in \( \mathcal{P}(p) \), and \( x \) is not on \( \text{Bd} \ N(p) \). So WLOG \( x \in \text{int}(\mathcal{K}_1') \). But (by Proposition 0) \( x \in \mathcal{U}\mathcal{P}' \) implies \( x \in \text{cl}(\mathcal{K}_3') \) and \( x \in \text{cl}(\mathcal{K}_4') \). Therefore \( \mathcal{K}_1 \setminus \mathcal{K}_1' \setminus \mathcal{U}\mathcal{P}(p) \) and \( K_2 \setminus \mathcal{K}_2' \setminus \mathcal{U}\mathcal{P}(p) \) are both non-empty. By the same type of argument \( \mathcal{K}_2 \setminus \mathcal{U}\mathcal{P}(p) \) is also non-empty. But if we pick one point from each of these sets then those three points must lie in distinct connected components of \( N(p) \setminus \mathcal{U}\mathcal{P}(p) \). So we are home by Proposition 1.

\[ \blacksquare \]

**DEFINITION**

We shall say that \( \mathcal{P} \) satisfies the \((\alpha, \beta)\)-form of condition \( 2 \) with respect to \((S,p)\) iff the following three sub-conditions all hold:

(a) If \( v \) is a vertex of two plates \( \pi_1, \pi_2 \in \mathcal{P}(p) \) and \( \nu \nu \) is an edge of \( \pi_1 \), then \( \nu \nu \) is an edge of \( \pi_2 \).

(b) If \( x \in S(p) \) and \( y \) is a \( \beta \)-neighbour of \( x \) that is contained in \( S(p) \) then the straight line segment \( xy \) either does not meet \( \mathcal{U}\mathcal{P}(p) \) or meets \( \mathcal{U}\mathcal{P}(p) \) only at \( x \).
(c) If $\mathcal{N}(p)$ and $e=vn$ is an edge of $\mathcal{N}$ then $v$ is $\alpha$-adjacent to $p$. 

The following proposition is probably the most important single result in this chapter:

**PROPOSITION 3**

Suppose $p$ is an $(\alpha, \beta)$ surface point of the set $S$, and suppose further that $S^c(p)$ has exactly two $\beta$-components. Let $P$ be a set of plates which is $\beta$-natural with respect to $(S, N(p))$ and which satisfies the $\beta$-form of condition 1 and the $(\alpha, \beta)$-form of condition 2 with respect to $(S, p)$. Then $P(p)$ is the plate-set of a single plate cycle at $p$, and every $\alpha$-neighbour of $p$ that is contained in $S$ is a vertex of some plate in $P(p)$.

**REMARK** The proof makes use of the fact that if $ab$ is an edge of a plate $\mathcal{N}$ then if $\epsilon$ is positive and sufficiently small $(B(a, \epsilon) \cap N(b)) \setminus \mathcal{N}$ is connected, (where $B(a, r)$ denotes the open ball of radius $\epsilon$ and centre $a$). The reader can either prove this or include an extra hypothesis in the statement of the proposition to the effect that all plates in $P$ have this property. (It is a trivial matter to check that every plate introduced in the next chapter satisfies this condition.)

**PROOF OF PROPOSITION 3**

Suppose the hypotheses are satisfied. Then $P(p)$ is $\beta$-natural with respect to $(S, N(p))$.

Let $v$ be $p$ or an $\alpha$-neighbour of $p$ that is contained in $S$: then by the definition of an $(\alpha, \beta)$ surface point there are $\beta$-neighbours $y_1, y_2$ of $v$ which are in different $\beta$-components of $S^c(p)$. So, by Proposition 1, $y_1$ and $y_2$ are in different connected components of $N(p) \setminus U(P(p))$. By condition 2(b) the straight line segments $y_1v$ and $y_2v$ can only meet $UP(p)$ at $v$. Hence $v \in UP(p)$, so $v$ is a vertex of a plate $\mathcal{N}$ in $P(p)$.

Further, if $vn$ is an edge of a plate then $v$ is a vertex of a second plate in $P(p)$. For suppose otherwise: then choose $r > 0$ such that $(B(v, r) \cap N(p)) \setminus \mathcal{N}$ is connected whenever $0 < \epsilon < r$ (see the remark above). Now choose $n(>1)$ so large that $(B(v, r/n) \setminus N(p)) \cup U(P(p)$ is connected, so there is a path in $P(p)$ except $\mathcal{N}$; then $B(v, r/n) \cap N(p) \cup U(P(p)$ is connected, so there is a path in
Next suppose \( e = xp \) is an edge of a plate \( \Pi \) in \( P(p) \). Then by condition 2(c) \( x \) is \( u \)-adjacent to \( p \), so by the previous paragraph \( x \) is a vertex of a plate \( \Pi' \) in \( P(p) \), where \( \Pi' \not= \Pi \). Condition 2(a) implies that \( xp \) is also an edge of \( \Pi' \). So we have shown that any edge of a plate in \( P(p) \) that has \( p \) as an endpoint is an edge of at least two plates in \( P(p) \). Now the definition of \( \beta \)-natural, condition 2(a) and the previous sentence together imply that \( P(p) \) satisfies the pre-conditions (i), (ii), (iii) of the cycle-finding algorithm. On applying this algorithm we get a plate-cycle at \( p \). Suppose for the purpose of getting a contradiction that there is a plate in \( P(p) \) which is not in this plate-cycle.

There may be two plates \( \Pi_0, \Pi_1 \) in \( P(p) \) such that \( \Pi_0 \) is in the cycle, \( \Pi_1 \) is not in the cycle, and \( \Pi_0 \) and \( \Pi_1 \) have an edge in common: if so then apply the cycle-finding algorithm to \( P(p) \) starting from \( (\Pi_0, \Pi_1) \). If two such plates do not exist then apply the cycle-finding algorithm to \( P(p) \) starting from any pair of plates which share an edge containing \( p \) but which are not in the cycle already found. In either case we obtain a plate-cycle that is different from the first one. Let \( P' \) and \( P'' \) be the sets of plates in the two cycles; then by condition 1(b) \( S^c(p) \) has at least three \( \beta \)-components, contrary to hypothesis. This contradiction proves Proposition 3. 

REMARK

Our plan is to show that for every \((a, B)\) surface-point \( p \) of \( S \) there exists a set of plates satisfying the hypotheses of Proposition 3. Proposition 3 will then give us a necessary geometrical condition for \( p \) to be an \((a, B)\) surface point of \( S \). We will use the following proposition to show that the necessary condition is also sufficient.
PROPOSITION 4

Suppose \( p \in S \subseteq \mathbb{Z}^3 \) and \( \mathcal{P} \) is the plate-set of a plate-cycle at \( p \) satisfying the hypotheses (i), (ii), (iii) of Proposition 2. Suppose further that \( D \) is a subset of \( N(p) \) such that \( \text{int}(D) \) meets \( \mathcal{U} \mathcal{P} \) and \( D \setminus \mathcal{U} \mathcal{P} \) has exactly two connected components \( C_1, C_2 \) both of which meet \( S^C(p) \). Then \( C_1 \cap S(p) \) and \( C_2 \cap S(p) \) are contained in different \( \beta \)-components of \( S(p) \).

PROOF

Suppose the conditions hold. Pick \( x \in \text{int}(D) \cap \mathcal{U} \mathcal{P} \); then by Proposition 2 \( x \in \text{cl}(K_1) \) and \( x \in \text{cl}(K_2) \), where \( K_1 \) and \( K_2 \) are the two connected components of \( N(p) \setminus \mathcal{U} \mathcal{P} \). So \( \text{int}(D) \cap K_1 \) and \( \text{int}(D) \cap K_2 \) are non-empty. Since \( D \) meets both \( K_1 \) and \( K_2 \) it is impossible for either one of \( K_1 \) and \( K_2 \) to contain both \( C_1 \) and \( C_2 \). So WLOG \( C_1 \subseteq K_1 \) and \( C_2 \subseteq K_2 \). The result now follows from Proposition 1. ■

Geometrical Characterizations of \((\alpha, \beta)\) Surface-Points

DEFINITION

If \( p \) is an \((\alpha, \beta)\) surface-point of \( W \) then \( A^\beta(p) \) and \( B^\beta(p) \) denote the two \( \beta \)-components of \( S^C(p) \) that are \( \beta \)-adjacent to \( p \). (We do not care which of the two \( \beta \)-components is \( A^\beta(p) \) and which is \( B(p) \): the purpose of this notation is merely to allow us to refer to each \( \beta \)-component separately.) ■

PROPOSITION 5

Let \( p \) be an \((\alpha, \beta)\) surface-point of \( S \) and let \( \mathcal{K} \) be a unit cell in \( N(p) \). In the cases where \( \beta = \mathcal{E} \) suppose further that each \( \alpha \)-neighbour of \( p \) that is contained in \( S \mathcal{N} \mathcal{K} \) is also an \((\alpha, \beta)\) surface-point of \( S \). Then \( S^C \mathcal{N} \mathcal{K} \) is \( \beta \)-adjacent to \( p \).
PROOF

Case I. \(\beta=18\) or 26

WLOG \(p = 111\) and \(K\) is the cell whose corners are \(\{xyz \mid x, y, z \in \{1, 2\}\}\). Suppose the result fails. Then \(Z21, 121, 12Z, Z12, Z11, 111, 11Z\) are in \(S\). As \(\{10Z, 20Z, 201, 101\}\) cannot meet both \(A^c(111)\) and \(B^c(111)\) we may assume WLOG that it does not meet \(A^c(111)\). Then by \((SP(S), Z^c(111), 11Z)\) one of \(002, 01Z, 02Z, 001, 011, 021\) is in \(A^c(111)\), and by \((SP(S), Z^c(111), 211)\) one of \(200, 210, 220, 100, 110, 120\) is in \(A^c(111)\). Therefore, since \(A^c(111)\) is \(\beta\)-connected one of \(001, 011, 021\) is in \(A^c(111)\), and one of \(100, 110, 120\) is in \(A^c(111)\). Hence none of the nine points in \(Z^c(111)\) with \(y = 1\) can be in \(B^c(111)\), so either \(y = 0\) for all points in \(B^c(111)\), or \(y = 2\) for all points in \(B^c(111)\). The former is impossible because \(121 \in S\) implies \(121\) is \(\beta\)-adjacent to \(B^c(111)\); the latter implies that \(101\) is neither in \(B^c(111)\) nor \(\beta\)-adjacent to \(B^c(111)\), whence \(101 \in A^c(111)\) contrary to our earlier assumption.

Case II. \(\beta=6\)

WLOG \(p=111\) and \(K\) is the same cell as before. Suppose the result fails. Then \(121, 11Z, Z11\) are \((u, 6)\) surface-points of \(S\). WLOG \(011 \in A^c(111)\) and \(101 \in B^c(111)\). Then by \((SP(S), Z^c(211), 111)\) \(110 \in S^c\), so WLOG \(110 \in B^c(111)\). Since \(011 \notin A^c(111)\) none of \(010, 021, 01Z\), \(001\) can be in \(B^c(111)\). Hence no \(S\)-path in \(B^c(111)\) from \(101\) to \(110\) can go through \(Z^c(111)\). So there is a \(S\)-path in \(B^c(111)\) from \(101\) to \(110\) which lies entirely within \(Z^c(111)\). This implies that \(101\) and \(110\) are in the same \(S\)-component of \(S^c(211)\), so \((SP(S), Z^c(211), 111)\) is violated.

COROLLARY

If \(p\) is an \((u, 6)\) surface-point of a set \(S\), where \(\beta = 18\) or 26 then \(S^c(p)\) has exactly two \(\beta\)-components, and both are \(\beta\)-adjacent to \(p\).
PROOF

The result is trivial if $\beta = 26$. Suppose $\beta = 18$, and $K$ is any unit cell in $N(p)$. By the above proposition there is some point $q$ in $KN^c$ such that $q$ is $18$-adjacent to $p$. Since every point in $KN^c$ is $18$-adjacent either to $p$ or to $q$, $KN^c$ is $18$-connected and $10$-adjacent to $p$. As $K$ is an arbitrary unit cell in $N(p)$, every $18$-component of $S^c(p)$ is $18$-adjacent to $p$. So we see that the Corollary follows from the definition of an $(\alpha, 10)$ surface-point. ■

(\alpha, 26) Surface-Points

DEFINITION

Let $S$ be any set of points in $\mathbb{Z}^d$ such that no unit cell contains eight points in $S$. Then we define $F(26,S)$ to be the set of $1 \times 1$ squares whose corners all lie in $S$. ■

We claim that if $p \in S$ then $F(26,S)$ satisfies the $26$-form of condition 1 and the $(\alpha, 26)$-form of condition 2 with respect to $(S,p)$ with $\alpha = 6, 18,$ or $26$. We further claim that $F(26,S)$ is $26$-natural with respect to $S$. It is easy to confirm the naturalness of $F(26,S)$ and the validity of condition 2. (In fact $F(26,S)$ is a continuous analogue of $(\alpha, 26,S)$, and not merely a weak analogue.) Furthermore, $F(26,S)$ satisfies the hypotheses (i), (ii), (iii) of Proposition 2 with $\beta = 26$, so $F(26,S)$ satisfies the $26$-form of condition 1 with respect to $(S,p)$.

PROPOSITION 6

If $p \in S \subseteq \mathbb{Z}^d$ then $p$ is an $(\alpha, 26)$ surface-point of $S$ iff the following all hold:
(i) No unit cell in $N(p)$ contains eight points in $S$.

(ii) $F(26,5\langle p \rangle \langle p \rangle)$ is the plate-set of a single plate-cycle at $p$.

(iii) If $q$ is an $\alpha$-neighbour of $p$ that is contained in $S$ then $q$ is a vertex of some plate in $F(26,5\langle p \rangle \langle p \rangle)$.

PROOF

"only if": (i) follows from Proposition 5. If $p$ is an $(\alpha,26)$ surface point of $S$ then $S^5\langle p \rangle$ has exactly two $\beta$-components, so (ii) and (iii) follow from Proposition 3.

"if": Suppose (i), (ii) and (iii) all hold. Then by (ii) and condition 1(a) $S^5\langle p \rangle$ has exactly two 26-components. Let $v$ be $p$ or any $\alpha$-neighbour of $p$ that is contained in $S$; by (iii) $v$ is a vertex of some plate $\Pi$ in $F(26,5\langle p \rangle \langle p \rangle)$. WLOG $p = 111$, and the vertices of $\Pi$ are $111, 121, 117, 112$. Let $D_1$ and $D_2$ denote the closed unit cells in $N(111)$ containing 022 and 222 respectively. Each of $D_1$ and $D_2$ meets $S^5\langle 111 \rangle$ by (i).

Further, $D_1 \setminus UF(26,5\langle 111 \rangle \langle 111 \rangle)$ and $D_2 \setminus UF(26,5\langle 111 \rangle \langle 111 \rangle)$ are both connected sets, so on applying Proposition 4 to $D_1 \cup D_2$ we deduce that $D_1 \cap S^5\langle 111 \rangle$ and $D_2 \cap S^5\langle 111 \rangle$ are contained in different 26-components of $S^5\langle 111 \rangle$. But every point in these two sets is 26-adjacent to each vertex of $\Pi$. So $p = 111$ is an $(\alpha,26)$ surface-point of $S$, as required.

COROLLARY

If $S \leq 2^\circ$ then $S$ is an $(\alpha,26)$ digital-surface iff (i), (ii) and (iii) hold for all $p$ in $S$. ■

$(\alpha,18)$ Surface-Points

DEFINITION

If $X$ is a unit cell, $g$ is the centroid of $X$, and $x, y$ are any two diametrically opposite corners of $X$, then the set $U\langle xuv \rangle \cup \{u, v\}$ are $6$-adjacent points in $X\setminus\{x, y\}$ will be called
a compound plate. 

Thus if \((S,K)\) is of type 10 then \(C^*(S,18.5^NK)\) is a compound plate. Figure 2 shows a compound plate.

**DEFINITION**

Let \(S\) be a subset of \(\mathbb{Z}^2\) such that no unit cell contains eight points in \(S\). Then we define \(F(18.S)\) to be the set of plates such that \(P \in F_{18}(S)\) iff one of the following is true:

either \(P\) is a 1x1 square whose corners are all in \(S\),
or \(P\) is a compound-plate whose vertices are all in \(S\) and the unit cell containing \(P\) contains no point in \(S\) that is not a vertex of \(P\).

We claim that if \(p \in S\) then \(F(18.S)\) satisfies the 10-form of condition 1 and the \((\alpha,18)\)-form of condition 2 with respect to \((S,p)\) with \(\alpha = 6, 18,\) or 26. We further claim that \(F(10.S)\) is 18-natural with respect to \(S\). (In fact \(F(18.S)\) is a continuous analogue of \((\alpha,18.S)\), and not merely a weak analogue.) It is easy to confirm the naturalness of \(F(18.S)\) and the validity of condition 2. To see that \(F(18.S)\) satisfies condition 1 with respect to \((S,p)\) define a function \(f\) on \(F(18.S)(p)\) as follows:

(i) If \(P\) is not a compound-plate then \(f(P) = P\)

(ii) If \(P\) is a compound-plate with vertices \(p, v_1, v_2, v_3, v_4, v_5\) where \(v_1\) is 6-adjacent to \(v_4\) and \(p\) is 6-adjacent to \(v_1\) and \(v_5\), then \(f(P) = (Apv_1v_3)U(Apv_5v_3)\)

Now the set \(f(F(18.S)(p))\) satisfies the hypotheses (i), (ii), (iii) of Proposition 2 with \(B=18\). So by Proposition 2 \(f(F(18.S)(p))\) satisfies the 18-form of condition 1 with respect to \((S,p)\). But if \(P' \in F(18.S)(p)\) and \(P'\) is the plate-set of a plate-cycle at \(p\) then \(f(P')\) is also the plate-set of a plate-cycle at \(p\). Hence \(F(18.S)\) also satisfies the 18-form of condition 1 with respect to \((S,p)\), as asserted.
FIGURE 2: A Compound Plate

FIGURE 3: A "Forbidden" Configuration
PROPOSITION 7

Suppose \( p \) is an \((0,18)\) surface-point of a set \( S \), and all \(0\)-neighbours of \( p \) that lie in \( S \) are also \((0,18)\) surface-points of \( S \). Then the following all hold:

(i) Each unit cell in \( N(p) \) contains at most six points in \( S \).

(ii) \( F(18,5<p>)\langle p \rangle \) is the plate-set of a single plate-cycle at \( p \).

(iii) If \( q \) is an \( n\)-neighbour of \( p \) that is contained in \( S \) then \( q \) is a vertex of some plate in \( F(18,5<p>)\langle p \rangle \).

Conversely, if \( p \in S \subseteq Z^3 \) and (i), (ii), (iii) all hold then \( p \) is an \((0,18)\) surface-point of \( S \).

PROOF

If \( p \in S \subseteq Z^3 \) and \( p \) and all its \(0\)-neighbours in \( S \) are \((0,18)\) surface-points of \( S \) then (i) follows from Proposition 5 while (ii) and (iii) follow from Proposition 5 (Corollary) and Proposition 3.

Conversely, suppose \( S \) satisfies (i), (ii) and (iii). Then WLOG \( p = 111 \). Then by (ii) and condition 1(a) \( S^5(111) \) has precisely two 18-components. Let \( v \) be 111 or an \(0\)-neighbour of 111 that lies in \( S \); then by (iii) there is a plate \( \Pi \) in \( F(18,5(111))\langle 111 \rangle \) that contains \( v \).

If \( \Pi \) is a compound-plate then WLOG the vertices of \( \Pi \) are 111, 211, 212, 222, 122 and 121. Let \( D \) be the closed unit cell of \( N(111) \) that contains 222. Then \( D \setminus \text{uf}(F(18,5(111))\langle 111 \rangle) \) has precisely two connected components, \( C_1 \) and \( C_2 \), say, and each of these contains one point in \( S^5 \). By Proposition 4 (applied to \( D \) and \( f(F(18,5(111))\langle 111 \rangle) \)), the points in \( C_1 \cdot S^5 \) and \( C_2 \cdot S^5 \) are in different 18-components of \( S^5(111) \). If, on the other hand, \( \Pi \) is a square plate then WLOG the vertices of \( \Pi \) are 111, 121, 122 and 112. Let \( D_1 \) and \( D_2 \) be the closed unit cells in \( N(111) \) that contain 222 and 022. Then by Proposition 4 \( D_1 \cdot S^5 \) and \( D_2 \cdot S^5 \) are contained in different 18-components of \( S^5(111) \). By (i) each of these two sets contains a point 18-adjacent to \( v \). So in each case \( v \) is 18-adjacent to both 18-components of \( S^5(111) \). Hence \( p = 111 \) is an \((0,18)\) surface-point of \( S \).
COROLLARY

If \( S \subseteq 2^3 \) then \( S \) is an \( (0,18) \) digital-surface iff (i), (ii) and (iii) hold for every \( p \) in \( S \).

(0,6) Surface-Points

PROPOSITION 8

Suppose \( p \) is an \((18,6)\) surface-point of a set \( S \) and \( K \) is a unit cell in \( N(p) \) such that every 18-neighbour of \( p \) that lies in \( SNK \) is also an \((18,6)\) surface point of \( S \). Then \( (S,K) \) is of type 2, 3, 5, 6, 8, 9, 11, 12 or 13.

PROOF

In this proof 'surface-point' means '\((18,6)\) surface-point of \( S \)'.

Suppose the hypotheses are satisfied. \( \text{WLOG} \ K \) is the the unit cell in \( N(111) \) that contains 222. We shall prove the result by showing that the following four situations cannot arise.

1. \( SNK \) contains four corners of a regular tetrahedron.
2. \( SNK \) contains more than four points.
3. \( SNK \) consists of three of the four corners of a \( 1\times \sqrt{2} \) rectangle.
4. \( SNK \) contains exactly two points, and these are diagonally opposite corners of a face of \( K \).

(Proposition 5 implies that if \( SNK \) contains exactly four points and (1) does not occur then \( (S,K) \) is of type 9, 11, 12 or 13.)

Suppose (1) occurs. \( \text{WLOG} \ 111, 212, 122, 221 \) are in \( S \). Then by Proposition 5 \( p \) must be one of these four points and so \( \text{WLOG} \ p = 111 \). Proposition 5 now implies that none of 112, 121 and 211 is in \( S \). Thus \( \text{WLOG} \ 112 \) and 211 are in the same 6-component of \( S^c \langle 111 \rangle \). But now \( \langle 102,201 \rangle \) either meets \( S^c \langle r \rangle \) does not meet \( S^c \), and in both cases 212 is 6-adjacent
to just one 6-component of $S^5(111)$ that is 6-adjacent to 111. #

Next, suppose that (2) occurs. Let us first eliminate the possibility that $K$ contains four points in $S$ which are the corners of a face of $K$; if so then by symmetry we may assume that 111, 211, 221, 121 and 112 are in $S$. By hypothesis either 221 is a surface-point, in which case $(SP(S),Z^2(221),111)$ is violated, or $p = 117$, in which case 111, 121 and 211 are also surface-points, so that Proposition 5 is contradicted at 111. Now consider the case in which $SK$ does not contain four corners of a unit square. Since (1) does not occur WLOG 111, 112, 212, 221 and 121 are all in $S$. By symmetry we may assume that $p$ is one of these five points. Then 111 is a surface-point, and one of 112 and 121 is also a surface-point. Assume WLOG that 121 is a surface-point. By $(SP(S),Z^2(121),112) \in S$. Hence 201 and 710 are in $S$ and 211 $\in S^j$ by $(SP(S),Z^2(111),212)$ and $(SP(S),Z^2(111),221)$. So $(211)$ is a 5-component of $S^5(111)$ that is not 6-adjacent to 112, whence 111 is not a surface-point. #

Suppose finally that (3) or (4) occurs. Then WLOG 111 and 212 are in $S$ and 112, 122, 221, 211 are all in $S^j$. By hypothesis 111 must be a surface-point. But now $(102,201)$ either meets $S^j$ or does not meet $S^i$, and in both cases 212 is 6-adjacent to just one 6-component of $S^5(111)$ that is 6-adjacent to 111. #

**PROPOSITION 5**

If $p$ is an $(18,6)$ surface-point of a set $S$ and every 18-neighbour of $p$ that lies in $S$ is also an $(10,6)$ surface-point of $S$ then each 6-component of $S^5(p)$ is 6-adjacent to $p$.

**PROOF**

Suppose the hypotheses are satisfied, but there is a 6-component of $S^5(p)$ which is not 6-adjacent to $p$. In view of Proposition 8, it is easy to see that this 6-component cannot contain any of the eight corners of $Z^2(p)$, so it must consist of a single point which is 18-adjacent but not 6-adjacent to $p$. Then WLOG $p = 011$ and $\{112\}$ is a 6-component of $S^5(011)$. This means that 012, 102, 122 and 111 are in $S$, so (by Proposition 8) 101, 001,
022, 121, 021 and 022 are in $S$.

Suppose WLOG that 112 $\in \mathcal{A}^c(111)$; then 212 $\in \mathcal{A}^c(111)$ (else 011 is not $S$-adjacent to
$\mathcal{A}^c(111)$, a contradiction), and therefore 022, 021 and 121 are in $\mathcal{B}(111)$ (for if 022, 021 and 121 are in $\mathcal{A}^c(111)$ then 122 is not $S$-adjacent to $\mathcal{B}(111)$, a contradiction). Similarly
002, 001 and 101 are in $\mathcal{B}(111)$. So by $(SP(5), 2^{x}1111, 0111, 0110, 0111) \in \mathcal{A}^c(111)$, whence there is
a $S$-path in $\mathcal{A}^c(111)$ from 010 to 212. Consequently 110 $\in \mathcal{A}^c(111)$, which implies that $\mathcal{B}(111)$
is not $S$-connected. This contradiction proves the proposition. ■

**COROLLARY 1**

If $p$ is an $(18, 6)$ surface-point of a set $S$ and every 18-neighbour of $p$ that lies in $S$ is
also an $(10, 6)$ surface-point of $S$ then $S(p)$ has exactly two $S$-components, and both
are $S$-adjacent to $p$. ■

**COROLLARY 2** (This is a corollary to the proof of Proposition 9.)

If $p$ is an $(18, 6)$ surface-point of a set $S$ and all 18-neighbours of $p$ that lie in $S$ are
also $(18, 6)$ surface-points of $S$ then no 2x1x1 cell in $N(p)$ is identical to the cell in
Figure 3.

**PROOF**

Suppose the hypotheses are satisfied and such a 2x1x1 cell exists. WLOG the cell
contains 002, 022 and 111. and WLOG 011, 012, 102, 122, 111 are in $S$. Then $p = 011, 012$
of 111 and so 111 must be an $(18, 6)$ surface-point of $S$. Hence the argument given in the
second paragraph of the proof of Proposition 9 produces the required contradiction. ■

**PROPOSITION 10**

If $p$ is a $(26, 6)$ surface-point of a set $S$ and every 26-neighbour of $p$ that lies in $S$ is
also a $(26, 6)$ surface-point of $S$ then every unit cell in $N(p)$ is of type 2, 3, 8, 9 or
13 relative to $S$. 

132
PROOF

Suppose the hypotheses are satisfied. Then WLOG $p = 111$. In virtue of Proposition 8 it suffices to establish the following two assertions:

(i) If two diametrically opposite corners of a unit cell $K$ in $N(p)$ are both in $S$ then $(K,S)$ is of type 13

(ii) There is no cell $K$ in $N(p)$ such that $(K,S)$ is of type 6

To prove (i), suppose 222 is in $S$. We claim this implies that the unit cell containing 111 and 222 is of type 13. For 222 must be 6-adjacent to $A^6(111)$ and to $B^6(111)$, so WLOG $127 \in A^6(111), 221 \in B^6(111)$ and $121 \in S$. Then, since 111 is 6-adjacent to two 6-components of $S^5(222)$, 112 and 211 are in $S^5$ and 212 is in $S$. So our claim is justified.

To prove (ii), suppose on the contrary that 111, 211, 212 are in $S$ and 112, 122, 222, 221 and 121 are in $S^5$. Then by (i) 101, 102 and 202 are in $S^5$, so 212 is 6-adjacent to only one 6-component of $S^5(111)$, a contradiction. $\blacksquare$

DEFINITION

Let $S$ be a subset of $\mathbb{Z}^3$ such that no unit cell contains more than four points in $S$ and every cell containing four points in $S$ is of type 9, 11, 12 or 13. Suppose further that no $2 \times 1 \times 1$ cell is identical to the $2 \times 1 \times 1$ cell in Figure 3. Then we define $F(6,S)$ to be the set of plates such that $\Pi \in F(6,S)$ iff one of the following ((a)-(d)) applies:

(a) $\Pi$ is the union of two triangles $\triangle ABC$ and $\triangle BCD$, where $A$, $B$, $C$ and $D$ are the four points in a cell of type 12 which are in $S$, and $BC=\sqrt{3}$

(b) $\Pi$ is a $1 \times \sqrt{2}$ rectangle whose corners are the four points in a cell of type 13 which are in $S$

(c) $\Pi$ is a $(\sqrt{2}, \sqrt{2}, \sqrt{2})$ triangle whose corners are the three points in a cell of type 8 which are in $S$

(d) $\Pi$ is a $1 \times 1$ square whose corners are the four points in a cell of type 9 which are in $S$
We claim that if \( p \in S \) then \( F(6, S) \) satisfies the 6-form of condition 1 and the 
\((u, 6)\)-form of condition 2 with respect to \((S, p)\), with \( \alpha = 14 \) or 26. We further claim 
that \( F(6, S) \) is \( \mathbb{E} \)-natural with respect to \( S \). (In fact \( F(6, S) \) is a continuous analogue of \( \mathbb{E}(0, 6, S) \), and not merely a weak analogue.) It is easy to confirm the naturalness of 
\( F(6, S) \) and the validity of condition 2 by inspection of Figure 1 in Chapter 2. But \( F(6, S) \) 
satisfies the hypotheses (i), (ii), (iii) of Proposition 2 with \( B = 6 \). Hence, by 
Proposition 2, \( F(6, S) \) satisfies the 6-form of condition 1 with respect to \((S, p)\).

**PROPOSITION 11**

Suppose \( p \) is an \((18, 6)\) surface-point of a set \( S \) and all 18-neighbours of \( p \) that lie in \( S \) 
are also \((18, 6)\) surface-points of \( S \). Then the following all hold:

(i) No unit cell in \( N(p) \) contains more than four points in \( S \) and every cell containing 
four points in \( S \) is of type 0, 11, 12 or 13 relative to \( S \).

(ii) No \( 2 \times 1 \times 1 \) cell in \( N(p) \) is as in Figure 4.

(iii) \( F(6, S(p)) \) is the plate-set of a single plate-cycle at \( p \).

(iv) If \( q \) is an 18-neighbour of \( p \) that lies in \( S \) then \( q \) is a vertex of some plate in

\( F(6, S(p)) \).

Conversely, if \( p \in S \subseteq Z^3 \) and (i), (ii), (iii) and (iv) all hold then \( p \) is an \((18, 6)\) 
surface-point of \( S \).

**PROOF**

If \( p \) is an \((18, 6)\) surface-point of a set \( S \) and all 18-neighbours of \( p \) that lie in \( S \) are 
also \((18, 6)\) surface-points of \( S \) then (i) and (ii) follow from Proposition 8 and 
Proposition 9 (Corollary 2), while (iii) and (iv) follow from Proposition 9 (Corollary 1) and 
Proposition 3.

Conversely, suppose \( p \in S \subseteq Z^3 \) and (i), (ii), (iii) and (iv) all hold. Then, since
F(6, S\langle p \rangle) satisfies the 6-form of condition 1 with respect to \( S, p \). (iii) implies that \( S^c(p) \) has exactly two 6-components. Let \( v \) be \( p \) or any 18-neighbour of \( p \) that lies in \( S \). Then by (iv) \( v \) is a vertex of a plate \( \Pi \) in \( F(6, S\langle p \rangle)\langle p \rangle \). We assert that \( v \) is 6-adjacent to both 6-components of \( S^c(p) \).

If \( \Pi \) is a plate of type (b) or (c) then this result can be obtained by applying Proposition 4 to \( F(6, S\langle p \rangle) \) (taking \( D \) to be the closed unit cell in \( N(p) \) containing the plate). If \( \Pi \) is of type (a) or (d), then WLOG \( p = 111 \), and by symmetry it is enough to establish the result in the following four cases:

I. The vertices of \( \Pi \) are 111, 212, 222 and 122
II. The vertices of \( \Pi \) are 111, 112, 222 and 121
III. The vertices of \( \Pi \) are 111, 112, 122 and 221
IV. The vertices of \( \Pi \) are 111, 121, 122 and 112

In each case the desired result can be deduced from Proposition 4 (applied to \( F(6, S\langle p \rangle)\langle p \rangle \)): in case I take \( D \) to be the closed unit cell in \( N(p) \) that contains 222; in the other cases take \( D \) to be the union of the closed unit cells in \( N(p) \) which contain 022 and 222 (note that by (i) and (ii) the interior of the cell containing 022 meets no plate in \( F(6, S\langle p \rangle) \) and that this cell always contains a point in \( S^c \) that is 6-adjacent to \( v \)). This argument justifies our assertion, which implies that \( p \) is an \((18, 6)\) surface point of \( S \).

**LUKULLARY**

If \( S \subseteq 2^d \) then \( S \) is an \((18, 6)\) digital-surface iff (i), (ii), (iii) and (iv) hold for all \( p \) in \( S \).
PROPOSITION 12

Suppose \( p \) is a \((26,6)\) surface-point of a set \( S \) and every 26-neighbour of \( p \) that lies in \( S \) is also a \((26,6)\) surface-point of \( S \). Then the following all hold:

(i) No unit cell in \( N(p) \) contains more than four points in \( S \), and every cell containing four points in \( S \) is of type 9 or 13 relative to \( S \).

(ii) \( F(S,5(p)) \) is the place-set of a single place-cycle at \( p \).

(iii) If \( q \) is a 26-neighbour of \( p \) that lies in \( S \) then \( q \) is a vertex of some place in \( F(S,5(p)) \).

Conversely, if \( p \in S \subseteq 2^\circ \) and (i), (ii) and (iii) all hold then \( p \) is a \((26,6)\) surface-point of \( S \).

PROOF

If \( p \) is a \((26,6)\) surface-point of a set \( S \) and every 26-neighbour of \( p \) that lies in \( S \) also a \((26,6)\) surface-point of \( S \) then (i) is Proposition 10 while (ii) and (iii) follow from Proposition 5 (Corollary 1) and Proposition 3.

Conversely, if \( p \in S \subseteq 2^\circ \) and (i), (ii) and (iii) all hold then by Proposition 11 \( p \) is an \((18,6)\) surface-point of \( S \). Suppose \( v \) is a 26-neighbour of \( p \) that lies in \( S \) and \( v \) is not 18-adjacent to \( p \). Then by (i) and (iii) \( v \) and \( p \) are diametrically opposite corners of a cell of type 13, so on applying Proposition 4 to this cell we deduce that \( v \) is 6-adjacent to two different 6-components of \( S^5(p) \), both of which are 6-adjacent to \( p \). Therefore \( p \) is a \((26,6)\) surface-point of \( S \). □

COROLLARY

If \( S \subseteq 2^\circ \) then \( S \) is a \((26,6)\) digital-surface iff (i), (ii) and (iii) all hold for every \( p \) in \( S \). □
A Jordan-Brouwer Separation Theorem, and Other Properties of Surface-Points

In the previous section we obtained simple 'visual interpretations' of eight of the nine different kinds of surface-point (the exception being the $(6,6)$ surface-points). We will now use these ideas to establish a number of basic results about surface-points and digital-surfaces. Special cases of Propositions 13 and 18 were proved by Morgenthaler, Reed and Rosenfeld in their papers. The complexity and subtlety of their arguments (which did not make use of continuous analogues) would seem to highlight the benefits of our approach.

PROPOSITION 13

Let $S$ be an $\alpha$-connected $(\alpha, \beta)$ digital-surface, where $\alpha$ and $\beta$ are not both equal to 6. Suppose $S \subseteq \text{nt}(Q)$ where $Q$ is a cuboid whose corners are all in $Z^3$ and whose edges are all parallel to the coordinate axes. Then $S \cap Q$ has exactly two $\beta$-components, and each $\beta$-component is $\beta$-adjacent to every point in $S$.

PROOF

Suppose the hypotheses are satisfied. Then, by Propositions 6(ii), 7(ii), 11(iii) and 12(ii), $UF(\beta, S)$ is a polyhedral quasi-surface (without boundary) contained in $\text{int}(Q)$. We claim that $UF(\beta, S)$ is strongly-connected. To see this, let $P$ be a maximal subset of $F(\beta, S)$ such that $UF$ is strongly-connected. Let $p$ and $q$ be any pair of $\alpha$-adjacent points in $S$. Then $F(\beta, S)[p] \cap F(\beta, S)[q]$ is non-empty, by Propositions 6(iii), 7(iii), 11(iv) and 12(iii). Furthermore, $q \in S$ implies $F(\beta, S)[q]$ is strongly-connected (this follows from Propositions 6(ii), 7(ii), 11(iii), 12(ii) and the fact that every plate-cycle is strongly-connected). Therefore if $F(\beta, S)[p] \subseteq P$ then $F(\beta, S)[q] \subseteq P$. Since $(p,q)$ is an arbitrary pair of $\alpha$-neighbours in $S$, and $S$ is $\alpha$-connected, it follows that $F(\beta, S)[r] \subseteq P$ for all $r$ in $S$. Hence $P = F(\beta, S)$, and so $UF(\beta, S)$ is strongly-connected.
Hence by Proposition 0 $\emptyset \cup F(\beta,S)$ has precisely two connected components, $C_1$ and $C_2$ say, and every point in $S$ is in the closure of both components. But $F(\beta,S)$ is $\beta$-natural with respect to $S$, so we deduce from sub-condition (b) of the definition of a weak analogue that each of $C_1$ and $C_2$ contains a point in $S^\alpha(p)$ for all $p$ in $S$. Furthermore, Proposition 1 implies that $C_1^\alpha S^\alpha$ and $C_2^\alpha S^\alpha$ are distinct $\beta$-components of $S^\alpha q$. Combining these two observations with Proposition 5 (Corollary) and Proposition 9 (Corollary 1) we get the required result. □

REMARK

The basic goal of [ReedRosenfeld82] and [Reed81] was to prove this proposition for $(6,26)$ and $(26,6)$ digital-surfaces. □

The following proposition is analogous to Proposition 2 in Chapter 3; it implies that the families $(\alpha,\beta,\ldots)$ are "F-regular" when at least one of $\alpha$ and $\beta$ is not equal to 6.

PROPOSITION 14

(i) Let $S$ be an $(\alpha,\beta)$ digital-surface, where $\beta = 18$ or 26 and let $p$ be any point in $S$. Let $W$ be the connected component of $U F(\beta,S)$ that contains $p$, and let $T = W \cap 2^\alpha$. Then $T$ is $6$-connected and $T$ is an $\alpha$-component of $S$. Furthermore, if $q \in T$ then $F(\beta,T) < q > = F(\beta,S) < q >$.

(ii) Let $S$ be an $(\alpha,6)$ digital-surface, where $\alpha = 18$ or 26, and let $p$ be any point in $S$. Let $W$ be the connected component of $U F(6,S)$ that contains $p$, and let $T = W \cap 2^\alpha$. Then $T$ is 18-connected and $T$ is an $\alpha$-component of $S$. Furthermore, if $q \in T$ then $F(6,T) < q > = F(6,S) < q >$.

PROOF

Let $P$ be the subset of $F(\beta,S)$ such that $U F = W$.
(i) Suppose the hypotheses are satisfied. If \( p \) and \( q \) are two points in \( \mathbb{W}^2 \) then (since \( \mathbb{W} \) is connected) there is a sequence \( \langle \Pi_i \rangle_{i \in \mathbb{N}} \) of plates in \( \mathbb{P} \) such that \( p \) is a vertex of \( \Pi_0 \), \( q \) is a vertex of \( \Pi_n \), and \( \Pi_i \) has a vertex in common with \( \Pi_{i+1} \) (if \( i < n \)).

But the set of vertices of each \( \Pi_i \) is \( 6 \)-connected, so there is a \( 6 \)-path in \( \mathbb{W}^2 \) from \( p \) to \( q \). Hence \( \mathbb{W}^2 \) is \( 6 \)-connected. \( T \) is an \( \alpha \)-component of \( S \), for if \( u \in T \) and \( v \) is a point in \( S \) that is \( \alpha \)-adjacent to \( u \) then there is a plate \( \Pi \) in \( F(\beta, S) \) that contains both \( u \) and \( v \) (by Propositions 6(iii) and 7(iii)); since \( u \in \mathbb{W} \) it follows that \( \Pi \subseteq \mathbb{W} \), which implies \( v \in T \).

Now let \( q \) be an arbitrary point in \( T \) and let \( K \) be any closed unit cell in \( N(q) \). If \( K \) contains a plate in \( F(\beta, S) \) then \( K \cap S \) must be \( \alpha \)-connected, so since \( T \) is an \( \alpha \)-component of \( S \) it follows that \( \mathbb{W}^2 \cap S = \mathbb{W} \cap T \), whence the sets \( \{ \Pi \in F(\beta, T) \mid \Pi \subseteq K \} \) and \( \{ \Pi \in F(\beta, S) \mid \Pi \subseteq K \} \) are the same. If \( K \) does not contain a plate in \( F(\beta, S) \) then since \( T \subseteq S \) it is easily seen that \( K \) contains no plate in \( F(\beta, T) \). So, since \( K \) is an arbitrary cell in \( N(q) \), \( F(\beta, S) < q > = F(\beta, T) < q > \), as asserted.

(ii) The proof is obtained from the proof of (i) by substituting '1\( G \)-connected' and '1\( G \)-path' for '6-connected' and '6-path', and invoking Propositions 11(iv) and 12(iii) instead of 6(iii) and 7(iii). \( \blacksquare \)

COROLLARY

Suppose \( \alpha \) and \( \beta \) are not both equal to 6. Then \( S \) is an \( (\alpha, \beta) \) digital-surface iff every \( \alpha \)-component of \( S \) is itself an \( (\alpha, \beta) \) digital-surface.

PROOF

Let \( T \) be an \( \alpha \)-component of an \( (\alpha, \beta) \) digital-surface \( S \), and let \( q \) be any point in \( T \).

Then by the proposition above \( F(\beta, T) < q > = F(\beta, S) < q > \), so since \( S \) is an \( (\alpha, \beta) \) digital-surface the 'only if' part is an immediate consequence of Propositions 6, 7, 11 or 12 (depending on the values of \( \alpha \) and \( \beta \)). To prove the 'if' part, suppose every
α-component of a set S is an (α, β) digital-surface. Then F(β, S) exists. For suppose there is a unit cell of a 'forbidden' type with respect to S (such as a cell with eight corners in S); then this cell meets S in an α-connected set and so it is also a 'forbidden' cell with respect to some α-component of S, a contradiction. Let q be any point in S, and let T be the α-component of S that contains q. Then F(β, S)(q) = F(β, T)(q), by the argument given in the second paragraph of the proof of part (i) of the proposition. The corollary now follows from Propositions 5, 7, 11 or 12 (depending on the values of α and β). □

REMARK
This corollary is the main reason why we did not require that an (α, β) digital-surface should be α-connected. □

Our next proposition shows that the corresponding result for (6, 6) digital-surfaces is false. (Our proof is essentially the same as the proof of Proposition 16 in [MorgenthalerKnoefeldR]).

PROPOSITION 15
A finite β-component of a (6, 6) digital-surface is never a (5, 6) digital-surface.

PROOF
Let T be a finite β-component of a (6, 6) digital-surface. Then since T is bounded we may assume WLOG that the z-coordinate of each point in T is positive, and WLOG 111∈T. Suppose T contains a point p whose z-coordinate is strictly greater than 1. Then since there is a 6-path in T from 111 to p there must be x, y such that (x, y, 1) and (x, y, 2) are both in T. Then (x, y, 1) is 6-adjacent to exactly one 6-component of S(x, y, 1), so T is not a (5, 6) digital-surface. This argument shows that if a 6-component of a (6, 6) digital-surface is itself a (6, 6) digital-surface then every point in that 6-component has the same z-coordinate; similarly every point in the 6-component has the same x-coordinate, and the same y-coordinate, whence the 6-component consists
of just one point, and so it is not a \((6,6)\) digital-surface. 

\[
\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}
\]

Plane \(N\), when \(N > 0\).

\[
\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

Plane 0.

FIGURE 4: A Non-Trivial 6-Connected \((6,6)\) Digital Surface -- Plane \(N\), \(N \geq 0\)

(The diagrams should be extended to infinity in the obvious way.)
FIGURE 4: A Non-Trivial 6-Connected (6,6) Digital Surface — Plane N, N < 0
(The diagrams should be extended to infinity in the obvious way.)

REMARK
If we omit the word "finite" in the statement of Proposition 15 then the result is obviously false, since if \( P \) is any coordinate plane then \( \mathbb{Z}^2 \setminus \mathbb{P} \) is a 6-connected (6,6) digital-surface. Non-trivial examples of 6-connected (6,6) digital-surfaces are not so easy to find. But they do exist, as Figure 4 shows.

PROPOSITION 16
(i) Suppose \( p \) is an \((\alpha, 6)\) surface-point of \( S \), where \( \alpha = 18 \) or \( 26 \). Then \( p \) is \( \alpha \)-adjacent to exactly one 6-component of \( S(p) \setminus \{p\} \).
(ii) Suppose \( p \) is an \((\alpha, 6)\) surface-point of \( S \), and every \( \alpha \)-neighbour of \( p \) that lies in \( S \) is also an \((\alpha, 6)\) surface-point of \( S \) where \( \alpha = 18 \) or \( 26 \). Then \( p \) is \( \alpha \)-adjacent to exactly one 16-component of \( S(p) \setminus \{p\} \).
PROOF

(i) Suppose the hypotheses are satisfied. Then \( F(\beta, S(p))\langle p \rangle \) exists and is the plate-set of a single plate-cycle at \( p \) by Proposition 3 and Proposition 5 (Corollary). But if \( \Pi \in F(\beta, S(p))\langle p \rangle \) then \( \Pi \) is either a square plate or a compound-plate, and in both cases \( \Pi \cap S\langle p \rangle \) is \( 6 \)-connected and \( 6 \)-adjacent to \( p \). So, since the intersection of the vertex-sets of two consecutive plates in a plate-cycle must contain a point in \( S\langle p \rangle \), \((U F(\beta, S(p))\langle p \rangle) \cap S\langle p \rangle \) is \( 6 \)-connected and \( 6 \)-adjacent to \( p \). By Proposition 3 and Proposition 5 (Corollary) every \( \alpha \)-neighbour of \( p \) that lies in \( S \) is contained in \((U F(\beta, S(p))\langle p \rangle) \cap S\langle p \rangle \), so \( p \) is \( \alpha \)-adjacent to just one \( 6 \)-component of \( S\langle p \rangle \backslash \langle p \rangle \), as asserted.

(ii) This is analogous to the proof of (i): the result follows from Proposition 3, Proposition 9 (Corollary 1), and the fact that if \( p \in S \) and \( \Pi \in F(\beta, S(p))\langle p \rangle \) then \( \Pi \cap S\langle p \rangle \) is \( 10 \)-connected and \( 10 \)-adjacent to \( p \). ■

COROLLARY

If the hypotheses of (i) or (ii) hold then \( p \) is \( \alpha \)-adjacent to exactly one \( \alpha \)-component of \( S\langle p \rangle \backslash \langle p \rangle \). ■

REMARK

All the earlier authors defined a 'simple surface-point' of a set \( S \) to be a point \( p \) such that \( p \) is an \((\alpha, \beta)\) surface-point of \( S \) and \( p \) is \( \alpha \)-adjacent to precisely one \( \alpha \)-component of \( S\langle p \rangle \backslash \langle p \rangle \). Proposition 15 (Corollary) shows that if one of \( \alpha \) and \( \beta \) is not equal to \( 6 \) then the extra axiom is unnecessary in the sense that it is automatically satisfied by every point of an \((\alpha, \beta)\) digital-surface. (If \( \alpha = \beta = 6 \) then the extra axiom is not redundant, as can be seen from any of the five central points in Figure 4.) ■
PROPOSITION 17

(i) The only \((18, 26)\) digital-surfaces are the sets \(Z^3nP\) where \(P\) is a coordinate plane.

(ii) If \(S\) is a \((26, 18)\) digital-surface which is not of the form \(Z^3nP\) for some coordinate plane \(P\) then \(F(18, S)\) contains only compound-plates.

PROOF

If an \((a, b)\) digital-surface \(S\) is a subset of a coordinate plane \(P\) then it is plain that \(S = Z^3nP\). So in proving (i) and (ii) we need only consider digital-surfaces that are not subsets of any coordinate plane.

Suppose \(S\) is an \((18, 26)\) digital-surface. Then \(F(26, S)\) is a polyhedral quasi-surface (without boundary). So if \(F(26, S)\) is not a subset of a coordinate plane there must be two plates \(\Pi_1\) and \(\Pi_2\) in \(F(26, S)\) such that \(\Pi_1\) and \(\Pi_2\) have an edge in common and \(\Pi_1\) and \(\Pi_2\) are perpendicular. WLOG the vertices of \(\Pi_1\) and \(\Pi_2\) are \((111, 121, 122, 112)\) and \((111, 121, 221, 211)\). Then by Proposition 6(iii) \(112\) and \(211\) are contained in a single plate in \(F(26, S)\) and so \(212 \in S\). Similarly \(222 \in S\). This contradicts Proposition 6(i) and so (i) is proved.

Now suppose \(S\) is a \((26, 18)\) digital-surface. Then \(F(18, S)\) is a polyhedral quasi-surface (without boundary). Suppose \(F(18, S)\) contains a square plate; then if \(F(18, S)\) is not a subset of a coordinate plane there are two plates \(\Pi_1\) and \(\Pi_2\) in \(F(18, S)\) such that \(\Pi_1\) and \(\Pi_2\) have an edge in common, \(\Pi_1\) is a square plate and either \(\Pi_2\) is a square plate perpendicular to \(\Pi_1\) or \(\Pi_2\) is a compound-plate. In the former case there are vertices \(x\), \(y\) of \(\Pi_1\) and \(\Pi_2\) respectively such that \(x\) and \(y\) are diametrically opposite corners of a unit cell. This contradicts Proposition 7(iii). In the latter case WLOG the vertices of \(\Pi_1\) are \((111, 121, 122, 112)\) and WLOG \(111\) and \(112\) are also vertices of \(\Pi_2\). Now one of \(012\), \(011\), \(212\) and \(211\) is a vertex of \(\Pi_2\), so WLOG \(012 \in S\). Then there is no plate in \(F(18, S)\) which contains both \(012\) and \(121\), and this contradiction to Proposition 7(iii) proves (ii).
COROLLARY

The only \((26,26)\) digital-surfaces are the sets \(Z^{26}_{v}P\) where \(P\) is a coordinate plane. 

PROPOSITION 18

Suppose \(Q\) is a closed cuboid whose corners are in \(Z^{3}\) and whose edges are parallel to the coordinate-axes. Let \(S\) be a subset of \(Z^{3}\) and let \(T\) be an \(a\)-component of \(S \cap \text{int}(Q)\) such that each point in \(T\) is an \((a, β)\) surface-point of \(S\), where \(β = 18\) or \(26\) (the same value of \(β\) is used for each point in \(T\)). Then there are two distinct \(β\)-components of \(S \cap \text{int}(Q)\) each of which is \(β\)-adjacent to every point in \(T\).

PROOF

Suppose the hypotheses are satisfied. Then \(T\) is \(α\)-connected. So by the definition of an \((α, β)\) surface-point any \(β\)-component of \(S \cap \text{int}(Q)\) that is \(β\)-adjacent to one point in \(T\) is \(β\)-adjacent to all points in \(T\). Thus it suffices to prove that each point in \(T\) is \(β\)-adjacent to two different \(β\)-components of \(S \cap \text{int}(Q)\).

Define \(P = \bigcup (F(β, S(t))) | t \in T\) and define \(Σ = UP\). (\(P\) exists by Proposition 5.) Then \(Σ\) is a polyhedral surface. By Proposition 3 and Proposition 5 (Corollary) no point in \(T\) can be on \(Σ\).

Now let \(Π\) be an arbitrary plate in \(P\). It is readily confirmed that \(nZ^n \cap \text{int}(Q)\) is \(6\)-connected regardless of which type of plate \(Π\) is and irrespective of the position of \(Π\) in \(Q\), so every vertex of \(Π\) is either in \(T\) or is on \(Σ\). Hence \((Σ \cap T) \setminus (Σ \cap Q)\) contains no vertices of \(Π\). But by definition of \(F\), \(Σ\) is a union of straight line segments each of which joins the \(6\)-adjacent vertices of a plate in \(P\). Hence \(Σ \cap T \subseteq Σ \cap Q\).

Let \(p\) be any point in \(T\). Then \(p \notin Σ \cap \text{int}(Q)\), so by Proposition 0 \(p\) is in the closure of two different connected components of \(Q \setminus Σ\). Call these connected components \(C_1\) and \(C_2\).

We know \(P\) is \(β\)-natural with respect to \((S, N(p))\) so, by sub-condition (b) in the definition
of a weak analogue, \( L_1 \cap S^5(p) \) and \( L_2 \cap S^5(p) \) are both non-empty. By Proposition 1 and Proposition 5 (Corollary) these two sets are distinct \( \beta \)-components of \( S^5(p) \) and they are both \( \beta \)-adjacent to \( p \). But each of \( L_1 \cap S^5 \) and \( L_2 \cap S^5 \) is a union of \( \beta \)-components of \( S^5 \cap i \), because it is easy to see that the straight line segment joining two \( \beta \)-adjacent points in \( S^5 \) cannot meet \( i \). So the result is proved. \( \blacksquare \)

Proposition 18 is a natural generalization of the main theorem of [ReedRosenfeld82]. The proposition may fail if we allow \( \beta \) to be equal to 6, because there can be a 'gap' between \( \text{Bd} \bigcup \{F(6,5,t) : t \in T\} \) and the surface of \( O \). A counterexample to Proposition 18 (with \( \beta = 6 \)) in which \( O \) is a cube with sides of length four and \( T \) contains the point at the centre of \( O \) is called a cross-cap (following Morgenthaler, Reed and Rosenfeld). [ReedRosenfeld82] contains one example, and Figure 5 shows another. (In Figure 5, \( O \) can be 5, 18 or 26. The points marked 1 are in \( S \).) Readers should have no difficulty at all in constructing their own cross-caps using the concepts discussed in this chapter.

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{array}
\]

Layer 1  Layer 2  Layer 3  Layer 4  Layer 5

**Figure 5**: A Cross-Cap

In this chapter we have said very little about the structure of \( (6,6) \) digital-surfaces. In fact \( (6,6) \) digital-surfaces are really quite unlike the other kinds of digital-surface, as Proposition 15 has already shown. A striking illustration of the strangeness of \( (6,6) \)
FIGURE 6: A (6,6) Global Cross-cap
surface-points is provided by the following example* of a (6,6) digital-surface whose complement in \( \mathbb{Z}^3 \) is 6-connected - we shall call it a global cross-cap. (Recall that by Proposition 13 and Proposition 14 (Corollary) the complement of an \((a,b)\) digital-surface cannot be 8-connected if \(a\) and \(b\) are not both equal to 6.) The global cross-cap referred to is a subset of \( \{(x,y,z) \in \mathbb{Z}^3 \mid |x| \leq 6, \; |y| \leq 6, \; |z| \leq 3\} \). In Figure 6 'Level n' is the set \( \{(x,y,n) \mid x \text{ and } y \text{ are in } \mathbb{Z}^3, \; |x| \leq 6, \; |y| \leq 6\} \). The points on the global cross-cap are labelled 'I'.

* The example was constructed by Dr. Roscoe.

Summary

Suppose \((a,b,c) \in \mathbb{W} \subseteq \mathbb{Z}^3\) and \(K = \{(x,y,z) \mid \max(|x-a|,|y-b|,|z-c|) < 1\}\). (So \(K\) is the interior of a cube with edges of length two.) In this chapter we have shown that if we use 6- and 26-connectivity for \(\mathbb{W}\) and \(\mathbb{W}^c\) respectively then \((a,b,c)\) is a 'simple surface point' of \(\mathbb{W}\) in the sense of Morgenthaler, Reed and Rosenfeld ([MorgenthalerRosenfeld81], [ReedRosenfeld82] and [Reed84]) if and only if the part of the continuous analogue \(C^6(\mathbb{Z},\mathbb{W}) \) of \(\mathbb{W}\) that lies in \(K\) is a surface (in the naive sense) which separates \(K\) into two regions.

This result was precisely stated in Proposition 6, and we have proved similar but slightly more complicated results (in Propositions 7, 11 and 12) for seven of the eight other varieties of 'simple surface point'. These results led us to a new and more 'intuitive' proof of the main theorems of Morgenthaler, Reed and Rosenfeld. (In fact Propositions 13 and 18 are more general than the results proved in [ReedRosenfeld82] and [Reed84].)

We have also discovered (see Proposition 16) that one of the three axioms used by the earlier authors to define these surface-points is not really necessary because it can be derived from the other two axioms - unless we use 8-connectivity both for \(S\) and for \(S^c\).
Appendix to Chapter 5: A Proof of Proposition 0

We shall need the following three-dimensional analogue of the Jordan Curve Theorem:

THE JORDAN-BROUWER SEPARATION THEOREM FOR A STRONGLY-CONNECTED POLYHEDRAL QUASI-SURFACE (WITHOUT BOUNDARY)

If \( S \) is a strongly-connected polyhedral surface without boundary then \( \mathbb{R}^3 \setminus S \) has precisely two components, and one of the components is bounded. \( S \) is the boundary of each component.

PROOF

This result is a corollary of the Alexander Duality Theorem, which is proved in many standard texts on Algebraic Topology (e.g., [Maunder69], Chapter 5). □

PROPOSITION 0

Let \( X \) be a closed cuboid in \( \mathbb{R}^3 \) and let \( S \) be a polyhedral surface contained in \( X \) such that \( \partial \text{Int}(X) \neq \emptyset \) and \( \partial S \subseteq \partial X \). Let \( p \) be any point in \( \partial \text{Int}(X) \). Then \( p \) is in the closure of two different connected components of \( X \setminus S \). Furthermore, if \( S \) is strongly-connected and \( \partial S = \partial \sigma \partial X \) then \( X \setminus S \) has exactly two connected components, and \( S \) is a subset of the closure of each of these two sets.

PROOF

We may assume that \( S \) is strongly-connected, for the result is certainly true if it holds for each 'strongly-connected component' of \( S \).

Suppose \( \partial S = \partial \sigma \partial (\partial X) \). Then either \( \partial S = \emptyset \) or \( \partial S = \sigma_1 u_2 \ldots \sigma_n \), where \( n \geq 1 \) and the \( \sigma_i \) are simple closed (polygonal) curves on \( \partial X \) such that \( \sigma_i \sigma_j \) contains only finitely many points whenever \( i \neq j \). Pick an arbitrary point \( q \) in \( (\partial X \setminus (\partial S \cup \partial S')) \). By the Jordan
Curve Theorem \((\partial X)\setminus \Sigma\) has exactly two components: say that a point \(x\) is outside \(X\), iff \(x\) is in the same component of \((\partial X)\setminus \Sigma\) as \(q\). Let \(C_1\) be the set of points which are outside an odd number of the \(\Sigma_i\), and let \(C_2\) be the set of points which are outside an even number of the \(\Sigma_i\) (if \(\partial X = \emptyset\) then \(C_1 = \emptyset\) and \(C_2 = \partial X\)). The Jordan Curve Theorem states that \(\Sigma_i\) is contained in the closure of both components of \((\partial X)\setminus \Sigma\). It follows that \(X \subseteq \text{cl}(C_1) \cap \text{cl}(C_2)\), and it is easy to see from this that \(C_1 \cup \Sigma\) is a strongly-connected polyhedral surface without boundary. So by the Jordan-Brouwer Separation Theorem \(\mathbb{R}^n \setminus (C_1 \cup \Sigma)\) has exactly two components, one bounded and one unbounded. Let \(B\) be the bounded component and let \(U\) be the unbounded component. It is clear that \(B\) is a subset of \(X\), and it is also clear that \(C_2\) is a subset of \(U\).

We claim that neither one of \(B \cup C_2\) and \(U \cup X\) can meet the closure of the other. This is plain if \(C_1 = \emptyset\), so suppose \(C_1 \neq \emptyset\) and pick \(x\) in \(C_1\). By the Jordan-Brouwer Separation Theorem \(x \in \text{cl}(B)\). Now pick \(\varepsilon\) so small that the open ball \(B(x, \varepsilon)\) does not meet \(\Sigma\). Let \((B(x, \varepsilon) \cup X) \setminus (C_1 \cup \Sigma)\) be connected and so must be entirely contained in \(B\). Hence \(x \notin \text{cl}(U \setminus X)\). Therefore \(C_1 \setminus \text{cl}(U \setminus X) = \emptyset\). As \(C_1 \cup \Sigma\) is a polyhedral surface it is closed, so \(\text{cl}(C_1) \setminus \text{cl}(U \setminus X) = \emptyset\).

We assert that \(C_2\) is contained in a single component of \(X \setminus \Sigma\). This is certainly the case if \(\Sigma \cap \partial X = \emptyset\), for then \(C_2 = \partial X\). If \(\Sigma \cap \partial X \neq \emptyset\) then \(C_2 \cup \Sigma\) is a strongly-connected polyhedral surface without boundary, so by the Jordan-Brouwer Separation Theorem \(\mathbb{R}^n \setminus (C_2 \cup \Sigma)\) has a bounded component \(B'\) and \(C_2\) is contained in \(\text{cl}(B')\); since \(B'\) must plainly be contained in \(X\) our assertion is proved.

Let \(x\) and \(y\) be any two points in \(U \setminus X\), and let \(w\) be any point in \(U \setminus X\). Then the paths in \(U\) from \(x\) and \(y\) to \(w\) must meet \(C_2\). Hence by the previous paragraph there is a path in \(U \setminus X\) from \(x\) to \(y\). This argument shows that \(U \setminus X\) is connected, and so \(X \setminus \Sigma\) has precisely two components, which are \(B \cup C_1\) and \(U \setminus X\). By the Jordan-Brouwer Separation Theorem \(\Sigma \cap \text{Int}(X) \subseteq \text{cl}(U) \setminus \text{Int}(X)\). By the Jordan Curve Theorem \(\partial X \subseteq \text{cl}(C_2) \subseteq \text{cl}(U) \setminus \text{Int}(X)\).
\text{cl}(\text{UNX}). \text{ Hence } \Sigma \subset \text{cl}(\text{UNX}). \text{ By the Jordan-Brouwer Separation Theorem } \Sigma \subset \text{cl}(B) \subset \text{cl}(B \cup C_i). \text{ So we have established the proposition in the case when } Bd \Sigma = \Sigma \cap Bd X.

It remains to consider the case in which \(\text{Bd } \Sigma\) is a proper subset of \(\Sigma \cap X\). Suppose WLOG that the centroid of \(X\) is the origin. Define a map \(f\) on \(\mathbb{R}^3\) such that \(f(x,y,z) = (2x,2y,2z)\). Let \(Y = f(X)\), and let \(L\) be the set whose members are the straight line segments joining each point on \(\text{Bd } \Sigma\) to the image of that point under \(f\). Let \(\Sigma' = \Sigma \cup UL\). Then it is easy to see that \(\Sigma'\) is a strongly-connected polyhedral surface and \(\text{Bd } \Sigma' = \Sigma' \cap \text{Bd } Y\). So by what we proved in the previous paragraph \(Y \setminus \text{Bd } \Sigma'\) has exactly two components, and if \(p\) is a point in \(\Sigma \cap \text{Int}(X)\) then \(p\) is in the closure of both components, whence \(p\) is in the closure of two different components of \(X \setminus \Sigma\). \(\blacksquare\)
CONCLUDING REMARKS

We have introduced a novel approach to digital topology, based on a generalization of the usual concept of a digital picture and the notion of a continuous analogue. This has yielded new results and generalizations of the work of previous authors, including some fundamental separation and connectivity properties of digital borders, and a discrete three-dimensional Jordan-Brouwer Separation Theorem, special cases of which were proved by Margenthaler, Reed and Rosenfeld using quite difficult combinatorial arguments. Standard continuous analogues have been defined for all reasonably well-behaved digital pictures, and the Euler characteristic of a digital picture has been defined to be the Euler characteristic of its standard analogue.

As is well-known, one proves that a proposed two-dimensional parallel thinning algorithm is topologically sound by establishing that each component of the input black set contains exactly one component of the skeleton and that each component of the complement of the skeleton contains exactly one component of the input white set. Unfortunately, there is no such simple characterization of topology preservation for three-dimensional parallel thinning algorithms. This thesis has proposed a precise and natural definition of what it means for a three-dimensional thinning algorithm to "preserve topology", which can be used to prove that an algorithm is topologically sound. We have carefully investigated the relationship of our approach to that of other researchers who have worked on this difficult problem.

Two possibilities for future research are suggested by this thesis. The first is the possibility of generalizing some of our results to three-dimensional non-rectangular lattices. Let $L$ be a locally finite subset of $\mathbb{R}^3$, and for each point $w$ in $L$, let $N(w; L)$ denote the the set of all points $x$ in $\mathbb{R}^3$ such that $w$ is a nearest neighbour of $x$ in $L$. Then $N(w; L)$ is a convex three-dimensional polyhedron. We say two points $w_1$ and $w_2$ in $L$ are $V$-adjacent, $E$-adjacent or $F$-adjacent according as $N(w_1; L)$ meets $N(w_2; L)$ in a single point, a straight-line segment or a two-dimensional polyhedron. (Of course, $V$, $E$ and $F$ stand for vertex, edge and face.) Applying these definitions to the usual rectangular...
lattice, we see at once that \( N(w; 2^3) \) is a unit cube for all lattice points \( w \), and so in this case \( V-, E- \) and \( F\)-adjacency are just \( 26-, 18- \) and \( 6\)-adjacency respectively. It might be interesting to see how much of our work can be generalized to arbitrary "well-behaved" lattices. Moreover, there are two non-rectangular lattices which may deserve special attention (the first of which has already been mentioned in Chapter 7).

The set \( L_1 = \{(x,y,z) \in \mathbb{Z}^3 | x+y+z \text{ is even}\} \) is often called the face-centred cubic lattice. It is readily confirmed that \( N(w; L_1) \) is a dodecahedral rhomboid for all \( w \) in \( L_1 \). One sees that \( E- \) and \( F\)-adjacency are the same for the face-centred cubic lattice, and that the \( F\)-neighbours (\( = E\)-neighbours) of \( w \) are just the twelve points closest to \( w \). 

\( F\)-adjacency was considered in Chapter 7 (and in [KongRoscoe85c]). The \( V\)-neighbours are the \( F\)-neighbours plus the six points whose distance from \( w \) is exactly two. Just as it is often appropriate to use different adjacencies for object and background points in the case of the rectangular lattice, one should consider using different adjacencies for object and background on the face-centred cubic lattice. That is, we should consider \( (V,F)\)-adjacency and \( (F,V)\)-adjacency in addition to \( V- \) and \( F\)-adjacency.

The set \( L_2 = \{(x,y,z) \in \mathbb{Z}^3 | x \equiv y \equiv z \pmod{2}\} \) is often called the body-centred cubic lattice. It is readily confirmed that \( N(w; L_2) \) is a truncated octahedron for all \( w \) in \( L_2 \). One sees that \( V-, E-, F\)-adjacency coincide for the body-centred cubic lattice, so we may refer to the \( \text{neighbours} \) of a point without ambiguity. Each point \( p \) has exactly fourteen neighbours; eight of these are at a distance of \( \sqrt{3} \) and the other six are at a distance of \( 2 \) from the point \( p \).

Dr. Roscoe has suggested two reasonable approaches to the digital topology of non-rectangular lattices. We can either attempt to convert them to a rectangular lattice by means of a "nice" homeomorphism (topology preserving map), or we can attempt to generalize the theory of continuous analogues to non-rectangular lattices.

We have so far done little work on the second approach, but we find that the first
approach looks quite promising for the face-centred cubic and body-centred cubic lattices. Indeed, for each of the five adjacency relations mentioned above there is an affine map which transforms it to a strongly-normal adjacency relation (i.e. a relation such that all digital pictures based on it are strongly normal) on the rectangular lattice. (In the case of the body-centred cubic lattice we get the adjacency relation considered in [Kovalevsky84].)

The other possibilities for future work concerns three-dimensional thinning algorithms. First, it would be good to find a satisfactory generalization of the main theorem of [Rosenfeld75] to three dimensions. As we have already mentioned, [Morgenthaler81] contains much interesting work on this problem. But in its present form Morgenthaler's main theorem on parallel deletion of north border points includes a hypothesis (concerning direction independence of the point deletion criteria) which might not always hold in practice. Moreover, as Morgenthaler pointed out, some of the arguments in [Morgenthaler81] depend quite heavily on intuitive reasoning. For these reasons we feel that there is scope for further research in this area. It may also be worthwhile to experiment with a more easily visualized definition of an end point.

Second, it may be possible to generalize the ideas on "thinning by contour tracing" in [Pavlidis80] and [Arcelli81] to three dimensions. As we suggested in Chapter 5, the three-dimensional analogue of a two-dimensional non-multiple pixel may be "an (x,R) surface point of the border which is 6-adjacent to an interior point" (or something quite similar). One might also try to find a three-dimensional version of the method used in [Arcelli81] to detect the "significant convexities" of the object by border analysis.
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