Topological adjacency relations on $\mathbb{Z}^n$

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Abstract

For which adjacency relations (i.e., irreflexive symmetric binary relations) $\mathcal{R}$ on $\mathbb{Z}^n$ does there exist a topology $\tau$ on $\mathbb{Z}^n$ such that the $\tau$-connected sets are exactly the $\mathcal{R}$-path-connected subsets of $\mathbb{Z}^n$? If such a topology exists then we say that the relation $\mathcal{R}$ is topological.

Let $l_1$ and $l_\infty$, respectively, denote the 4- and the 8-adjacency relations on $\mathbb{Z}^2$ and the analogs of these two relations on $\mathbb{Z}^n$ (for any positive integer $n$). Consider adjacency relations $\mathcal{R}$ on $\mathbb{Z}^n$ such that

1. For $x, y \in \mathbb{Z}^n$, $x l_1 y \Rightarrow x \mathcal{R} y \Rightarrow x l_\infty y$.
2. For all $x \in \mathbb{Z}^n$, the set $\{x\} \cup \{y | x \mathcal{R} y\}$ is $l_1$-path-connected.

Among the uncountably many adjacency relations $\mathcal{R}$ satisfying conditions 1 and 2 above, Eckhardt and Latecki showed that there are (up to isomorphism) just two topological relations on $\mathbb{Z}^2$, and essentially showed that there are just four topological relations on $\mathbb{Z}^3$.

We show in this paper that for any positive integer $n$ there are only finitely many topological adjacency relations on $\mathbb{Z}^n$ that satisfy conditions 1 and 2, and we relate the problem of finding these relations to the problem of finding all sets of vertices of an $n$-cube such that no two vertices in the set are the endpoints of an edge of the $n$-cube. From our main theorems we deduce the above-mentioned results of Latecki and Eckhardt, and also deduce that there are (again, up to isomorphism) exactly 16 topological adjacency relations on $\mathbb{Z}^4$ that satisfy conditions 1 and 2.

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1. Introduction

Let $l_1$ and $l_\infty$, respectively, denote the 4- and the 8-adjacency relations on $\mathbb{Z}^2$, and the analogs of these two relations on $\mathbb{Z}^n$ for any positive integer $n$. (Here $\mathbb{Z}^n$ denotes the set of all points with integer coordinates in $n$-space.) More precisely, for $p = 1$ and...
\[ \infty, x \|_p y \iff \| x - y \|_p = 1, \] where \( \| v \|_1 \) and \( \| v \|_\infty \), respectively, denote the sum and the maximum of the absolute values of the components of the vector \( v \). As an example, on \( \mathbb{Z}^3 \) the \( l_1 \) and the \( l_\infty \) relations are, respectively, the 6- and the 26-adjacency relations.

It is quite well known that there is a topology on \( \mathbb{Z}^n \) whose connected sets are exactly the \( l_1 \)-path-connected subsets of \( \mathbb{Z}^n \) (see, e.g., [3, Theorem 4.2.5]), but there is no topology whose connected sets are exactly the \( l_\infty \)-path-connected subsets of \( \mathbb{Z}^n \) (see [1,8] and [3, p. 90]). For which adjacencies \( \alpha \) on \( \mathbb{Z}^n \) does there exist a topology \( \tau \) on \( \mathbb{Z}^n \) such that the \( \tau \)-connected sets are exactly the \( \alpha \)-path-connected subsets of \( \mathbb{Z}^n \)? (Here and elsewhere in this paper we use the term adjacency to mean an irreflexive symmetric binary relation.) If such a topology \( \tau \) exists then we say that the adjacency \( \alpha \) is topological.

This paper deals with the problem of finding all topological adjacencies \( \alpha \) on \( \mathbb{Z}^n \) that have the following two properties:

1. For \( x, y \in \mathbb{Z}^n \), \( x \|_1 y \Rightarrow x \|_\infty y \).
2. For all \( x \in \mathbb{Z}^n \), the \( \alpha \)-neighborhood of \( x \) (i.e., the set \( \{ x \} \cup \{ y \mid x \|_\alpha y \} \) is \( l_1 \)-path-connected.

We say that an adjacency \( \alpha \) on \( \mathbb{Z}^n \) is admissible if it satisfies condition 1, and say that \( \alpha \) is \( l_1 \)-connected if it satisfies condition 2. When \( n \leq 2 \) every admissible adjacency is \( l_1 \)-connected.

Admissible and \( l_1 \)-connected adjacencies have previously been considered by Herman (see, e.g., [3]). The concept of an admissible adjacency on \( \mathbb{Z}^n \) is equivalent to Herman’s concept of a local spel adjacency of the digital space \( (\mathbb{Z}^n, l_1) \). The concept of an \( l_1 \)-connected admissible adjacency on \( \mathbb{Z}^n \) is equivalent to Herman’s concept of a very tight local spel adjacency of \( (\mathbb{Z}^n, l_1) \).

The problem of finding all admissible \( l_1 \)-connected topological adjacencies on \( \mathbb{Z}^n \) was essentially solved for \( n \leq 3 \) by Eckhardt and Latecki [2]: There are just two non-isomorphic adjacencies of this kind when \( n = 2 \), and just four when \( n = 3 \). (We say that the relations \( \alpha_1 \) and \( \alpha_2 \) on \( \mathbb{Z}^n \) are isomorphic if there is a bijection \( \sigma: \mathbb{Z}^n \rightarrow \mathbb{Z}^n \) such that \( x \alpha_1 y \iff \sigma(x) \alpha_2 \sigma(y) \). It will be shown in Section 5 that if \( \alpha_1 \) and \( \alpha_2 \) are admissible \( l_1 \)-connected topological adjacencies, then any such bijection \( \sigma \) must map \( l_1 \)-neighbors to \( l_1 \)-neighbors, and must therefore be an isometry with respect to the usual Euclidean metric.)

These adjacencies are shown in Figs. 1 and 2. The adjacencies shown in the right-hand diagram of Fig. 1 and the second diagram (from the left) of Fig. 2 have been considered in the context of digital images by Khalimsky and others (see, e.g., [5–7]).

We use a new approach to attack the problem for all positive integers \( n \). We show that for any positive integer \( n \) there are only finitely many admissible \( l_1 \)-connected topological adjacencies on \( \mathbb{Z}^n \), and relate the problem of finding these adjacencies to the problem of finding all sets of vertices of an \( n \)-cube such that no two vertices in the set are the endpoints of an edge of the \( n \)-cube. We show how the results of Latecki and Eckhardt for \( n = 2 \) and 3 can be obtained using our method, and solve the problem in the case \( n = 4 \). There are just 16 non-isomorphic admissible \( l_1 \)-connected topological adjacencies on \( \mathbb{Z}^4 \).
Fig. 1. The admissible topological adjacencies on $\mathbb{Z}^2$. Eckhardt and Latecki [2] showed that there are no other such adjacencies. This result will be deduced in Section 7.7 from our theorems. (All admissible adjacencies on $\mathbb{Z}^2$ are $l_1$-connected.)

Fig. 2. The admissible $l_1$-connected topological adjacencies on $\mathbb{Z}^3$. Four adjacencies on $\{0, 1\}^3$ are shown. Each adjacency $\alpha$ on $\{0, 1\}^3$ induces an adjacency $\alpha_{\mathbb{Z}}$ on $\mathbb{Z}^3$ such that, for $x, y \in \mathbb{Z}^3$, $x \alpha_{\mathbb{Z}} y$ if and only if $x \cdot y$ and $(x \mod 2) \alpha (y \mod 2)$. [For $p = (p_1, p_2, p_3) \in \mathbb{Z}^3$, $(p \mod 2)$ denotes the point $(p_1 \mod 2, p_2 \mod 2, p_3 \mod 2)$ in $\{0, 1\}^3$.] The four adjacencies on $\mathbb{Z}^3$ induced in this way by the adjacencies on $\{0, 1\}^3$ shown in this figure are admissible $l_1$-connected topological adjacencies, and every admissible $l_1$-connected topological adjacency on $\mathbb{Z}^3$ is isomorphic to one of these four. This result (which is essentially due to Latecki and Eckhardt [2]) will be deduced in Section 7.7 from our theorems.

2. Topological adjacencies

2.1. Terminology and notation

Let $\alpha$ be an adjacency on a set $D$. (As mentioned in the Introduction, we use the term adjacency to mean an irreflexive symmetric binary relation.) If $x, y \in D$ and $x \alpha y$ then we say that $y$ is $\alpha$-adjacent to $x$ and that $y$ is an $\alpha$-neighbor of $x$. An $\alpha$-path is a non-empty sequence $\langle q_0, q_1, \ldots, q_m \rangle$ such that $q_i \alpha q_{i+1}$ ($0 \leq i < m$); this is called an $\alpha$-path from $q_0$ to $q_m$, and the integer $m \geq 0$ is its length. If $\pi = \langle q_0, q_1, \ldots, q_m \rangle$ is any $\alpha$-path then the $\alpha$-path $\langle q_m, q_{m-1}, \ldots, q_0 \rangle$ is called the reverse of $\pi$ and is denoted by $\pi^{-1}$.

If $\pi_1 = \langle q_0, q_1, \ldots, q_m \rangle$ and $\pi_2 = \langle q_m, q_{m+1}, \ldots, q_{m+k} \rangle$ are $\alpha$-paths such that the final point of $\pi_1$ is the initial point of $\pi_2$, then the catenation of $\pi_1$ and $\pi_2$ is the $\alpha$-path $\langle q_0, q_1, \ldots, q_m, q_{m+1}, \ldots, q_{m+k} \rangle$; this $\alpha$-path is denoted by $\pi_1 \cdot \pi_2$. Here $m$ or $k$ may be zero: If one of $\pi_1$ and $\pi_2$ has length 0 then $\pi_1 \cdot \pi_2$ is equal to the other. If the final point of $\pi_1$ is not the same as the initial point of $\pi_2$ then $\pi_1 \cdot \pi_2$ is undefined. Note that $\cdot$ is associative on $\alpha$-paths in the sense that $(\pi_1 \cdot \pi_2) \cdot \pi_3$ and $\pi_1 \cdot (\pi_2 \cdot \pi_3)$ either are both undefined or are equal.
An \( \alpha \)-path \( \langle q_0, q_1, \ldots, q_m \rangle \) is said to be closed if \( q_0 = q_m \). A closed \( \alpha \)-path \( \pi = \langle p_0, p_1, \ldots, p_m = p_0 \rangle \) is a cyclic permutation of a closed \( \alpha \)-path \( \pi' = \langle q_0, q_1, \ldots, q_m = q_0 \rangle \), if \( p_0, p_1, \ldots, p_{m-1} \) is a cyclic permutation of \( q_0, q_1, \ldots, q_{m-1} \).

A set \( S \subseteq D \) is said to be \( \alpha \)-path connected, or, more briefly, \( \alpha \)-connected, if for all \( s, s' \in S \) there exists an \( \alpha \)-path \( \langle s = q_0, q_1, \ldots, q_m = s' \rangle \) in which each \( q \) belongs to \( S \). (This \( \alpha \)-path may be of length 0, so all singleton subsets of \( D \) are \( \alpha \)-connected. Notice also that the empty set is \( \alpha \)-connected.) The \( \alpha \)-connected sets of size 2 completely determine \( \alpha \), since \( x \alpha y \) if and only if \( x \neq y \) and \( \{x, y\} \) is \( \alpha \)-connected.

If there exists a topology \( \tau \) on \( D \) such that, for every subset \( S \subseteq D \), \( S \) is \( \alpha \)-connected if and only if \( S \) is \( \tau \)-connected, then we say that the adjacency \( \alpha \) is topological; any such topology \( \tau \) will be said to generate \( \alpha \).

2.2. Transitive orientations of adjacencies: a characterization of topological adjacencies

For any adjacency \( \alpha \), a strict partial order \( \nearrow \) such that
\[ x \alpha y \iff x \nearrow y \text{ or } y \nearrow x \]
will be called a transitive orientation of \( \alpha \). Our work is based on the following fairly well-known characterization of topological adjacencies:

**Theorem 2.1.** An adjacency \( \alpha \) is topological if and only if there exists a transitive orientation of \( \alpha \).

Equivalently, an adjacency \( \alpha \) on a set \( D \) is topological if and only if the undirected graph induced by \( \alpha \) on \( D \) is a comparability graph. (There is a well-developed theory of such graphs (see, e.g., [4, 9]).)

In the rest of this section we outline a proof of Theorem 2.1; topologists will have little difficulty in supplying the missing details. (For a more complete proof of Theorem 2.1, see [10].) This is the only part of the paper that uses topological arguments.

For any topology \( \tau \) on a set \( D \) the specialization relation of \( \tau \), which we denote by \( \nearrow_{\tau} \), is the relation on \( D \) defined by \( x \nearrow_{\tau} y \iff x \in \text{cl} \{y\} \). Note that \( x \nearrow_{\tau} y \) if and only if every \( \tau \)-open set that contains \( x \) also contains \( y \). Evidently, \( \nearrow_{\tau} \) is reflexive and transitive. It is readily confirmed that if \( \tau \) is a topology that generates an adjacency, and \( \alpha \) is the adjacency generated by \( \tau \), then (since \( x \alpha y \) if and only if \( x \neq y \) and \( \{x, y\} \) is \( \tau \)-connected), \( x \alpha y \iff x \neq y \) and \( (x \nearrow_{\tau} y \text{ or } y \nearrow_{\tau} x) \). The following lemma is an immediate consequence of this fact:

**Lemma 2.2.** Let \( \alpha \) be an adjacency on a set \( D \). Then if \( \alpha \) is topological there exists a transitive relation \( \nearrow \) on \( D \) such that
\[ x \alpha y \iff x \neq y \text{ and } (x \nearrow y \text{ or } y \nearrow x). \]

The converse of this lemma is also true:
Lemma 2.3. Let $\sim$ be an adjacency on a set $D$. Then $\sim$ is topological if there exists a transitive relation $\nearrow$ on $D$ such that

$$x \sim y \iff x \neq y \text{ and } (x \nearrow y \text{ or } y \nearrow x).$$

Lemma 2.3 can be proved by verifying that if such a transitive relation $\nearrow$ exists, then $\sim$ is generated by the strongest topology on $D$ whose specialization relation is the reflexive closure of $\nearrow$. This is the topology whose open sets are the subsets $S$ of $D$ such that $\nexists s, t \in D \ (s \nearrow t \text{ and } s \in S \text{ and } t \notin S)$.

Theorem 2.1 follows from these two lemmas and the following lemma:

Lemma 2.4. Let $\nearrow$ be any transitive relation on a set $D$. Then there is a strict partial order $\nearrow^*$ on $D$ such that

$$x \nearrow^* y \text{ or } y \nearrow^* x \iff x \neq y \text{ and } (x \nearrow y \text{ or } y \nearrow x)$$

This lemma can be proved by verifying that if $<$ is any strict total order on $D$, then the relation $\nearrow^*$ on $D$ defined by

$$x \nearrow^* y \iff x \nearrow y \text{ and } (\text{not } (y \nearrow x) \text{ or } y < x)$$

is a strict partial order with the stated property.

3. Admissible functions

We say that a function $\theta : \{0, 1\}^n \to \mathbb{Z}$ is admissible if $\theta$ satisfies the following two conditions:

(A) Whenever $x \parallel y$, $|\theta(y) - \theta(x)| = 1$.

(B) $\theta(O) = 0$, where $O$ denotes the origin $(0, 0, \ldots, 0)$.

If $\theta$ is admissible then we write $\bar{\theta}$ for the function $-\theta$. Evidently, $\bar{\theta}$ is also an admissible function. We call each of $\theta$ and $\bar{\theta}$ the complement of the other, and call the unordered pair $\{0, \bar{\theta}\}$ a complementary pair of admissible functions.

Our main theorems (see Section 4) will show that there is a natural bijection from the set of admissible $l_1$-connected topological adjacencies on $\mathbb{Z}^n$ onto the set of complementary pairs of admissible functions $\theta, \bar{\theta} : \{0, 1\}^n \to \mathbb{Z}$. (It follows that for any positive integer $n$ there are only finitely many admissible $l_1$-connected topological adjacencies on $\mathbb{Z}^n$.) This bijection can be used to reduce the problem of finding all admissible $l_1$-connected topological adjacencies on $\mathbb{Z}^n$ to the considerably easier problem of finding all admissible functions on $\{0, 1\}^n$.

3.1. $\theta$-monotonic and $\theta$-increasing paths in $\mathbb{Z}^n$

For any point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$, we write $x \mod 2$ to denote the point $(x_1 \mod 2, x_2 \mod 2, \ldots, x_n \mod 2)$. 

Let \( \theta \) be any admissible function on \( \{0,1\}^n \), and let \( \pi = \langle q_0, q_1, \ldots, q_m \rangle \) be any \( l_1 \)-path in \( \mathbb{Z}^n \). We say that \( \pi \) is \( \theta \)-increasing if \( \langle \theta(q_i \mod 2) \mid 0 \leq i \leq m \rangle \) is an increasing sequence. We say that \( \pi \) is \( \theta \)-monotonic if \( \pi \) or \( \pi^{-1} \) is \( \theta \)-increasing.

Since \( \pi = \langle q_0, q_1, \ldots, q_m \rangle \) is an \( l_1 \)-path,

\[
m = \|q_m - q_0\|_1 \geq \|(q_m \mod 2) - (q_0 \mod 2)\|_1. \tag{1}
\]

For \( 1 \leq i \leq m \), since \( (q_{i-1} \mod 2) \) \( l_1 \) \( (q_i \mod 2) \) and since \( \theta \) is an admissible function, we have \( |\theta(q_i \mod 2) - \theta(q_{i-1} \mod 2)| = 1 \). Hence \( \pi \) is \( \theta \)-increasing if and only if \( \theta(q_i \mod 2) - \theta(q_{i-1} \mod 2) = 1 \) for \( 1 \leq i \leq m \). Thus \( \pi \) is \( \theta \)-increasing if and only if

\[
\theta(q_m \mod 2) - \theta(q_0 \mod 2) = \sum_{i=1}^{m} (\theta(q_i \mod 2) - \theta(q_{i-1} \mod 2)) = m. \tag{2}
\]

Moreover, regardless of whether or not \( \pi \) is \( \theta \)-increasing,

\[
|\theta(q_m \mod 2) - \theta(q_0 \mod 2)| \leq \sum_{i=1}^{m} |\theta(q_i \mod 2) - \theta(q_{i-1} \mod 2)| = m. \tag{3}
\]

For any \( x, y \in \mathbb{Z}^n \), on taking \( \pi \) to be a shortest \( l_1 \)-path from \((x \mod 2)\) to \((y \mod 2)\), so that \( m = \|(y \mod 2) - (x \mod 2)\|_1 \), (3) implies:

\[
|\theta(y \mod 2) - \theta(x \mod 2)| \leq \|(y \mod 2) - (x \mod 2)\|_1. \tag{4}
\]

Using (1), (2) and (4), we now establish a rather fundamental property of admissible functions that will be used in Section 4.

**Proposition 3.1.** Let \( \theta : \{0,1\}^n \rightarrow \mathbb{Z} \) be an admissible function and let \( x, y \in \mathbb{Z}^n \). Then \( \theta(y \mod 2) - \theta(x \mod 2) = \|y - x\|_1 \) if and only if there is a \( \theta \)-increasing \( l_1 \)-path from \( x \) to \( y \).

**Proof.** Suppose there exists a \( \theta \)-increasing \( l_1 \)-path \( \langle x = q_0, q_1, \ldots, q_m = y \rangle \) from \( x \) to \( y \). Combining (1) with (2), we get

\[
\theta(y \mod 2) - \theta(x \mod 2) \geq \|y - x\|_1 \geq \|(y \mod 2) - (x \mod 2)\|_1
\]

and hence, by (4), \( \theta(y \mod 2) - \theta(x \mod 2) = \|y - x\|_1 \).

Conversely, suppose \( \theta(y \mod 2) - \theta(x \mod 2) = \|y - x\|_1 \). Let \( \pi = \langle x = q_0, q_1, \ldots, q_m = y \rangle \) be a shortest \( l_1 \)-path from \( x \) to \( y \). Then \( m = \|y - x\|_1 = \theta(y \mod 2) - \theta(x \mod 2) \). Thus (2) holds and so \( \pi \) is \( \theta \)-increasing. \( \square \)

**Corollary 3.2.** Let \( \theta : \{0,1\}^n \rightarrow \mathbb{Z} \) be an admissible function and let \( x, y \in \mathbb{Z}^n \). Then \( |\theta(y \mod 2) - \theta(x \mod 2)| = \|y - x\|_1 \) if and only if there is a \( \theta \)-monotonic \( l_1 \)-path from \( x \) to \( y \).

3.2. The function \( h_\mathcal{A} : \mathbb{Z}^n \rightarrow \mathbb{Z} \)

In Sections 3.2 and 3.3, \( \mathcal{A} \) will denote an arbitrary admissible topological adjacency on \( \mathbb{Z}^n \), and \( \mathcal{A} \) will denote a transitive orientation of \( \mathcal{A} \). (As \( \mathcal{A} \) is topological, the
existence of a transitive orientation of $\pi$ is guaranteed by Theorem 2.1.) Later on, in Section 3.4, we will further assume that $\pi$ is $l_1$-connected.

In the present section we define a function $h_{\rightarrow} : \mathbb{Z}^n \to \mathbb{Z}$ whose restriction to $\{0,1\}^n$ is an admissible function. If $\pi$ is $l_1$-connected, this admissible function and its complement constitute the complementary pair of admissible functions that is the image of the adjacency $\pi$ under the bijection referred to above. The definition of the function $h_{\rightarrow} : \mathbb{Z}^n \to \mathbb{Z}$ will be based on an auxiliary function, also called $h_{\rightarrow}$, that maps $l_1$-paths in $\mathbb{Z}^n$ to integers.

We will say that a digraph matches a binary relation $\rho$ if $p \rho q$ is true whenever the digraph has an edge from $p$ to $q$. (Note that while a digraph that matches $\rho$ will not have an edge from $p$ to $q$ if $p \rho q$ is false, it may or may not have an edge from $p$ to $q$ if $p \rho q$ is true.)

A unit square path in $\mathbb{Z}^n$ is a closed $l_1$-path of length 4 that passes through four distinct points of $\mathbb{Z}^n$. Evidently, the reverse of a unit square path is a unit square path, and any cyclic permutation of a unit square path is a unit square path.

Let $\pi = (p_0, p_1, p_2, p_3, p_0)$ be any unit square path in $\mathbb{Z}^n$. If $\rho$ is any binary relation on $\mathbb{Z}^n$ such that $x \ l_1 y \Rightarrow x \rho y$ or $y \rho x$, then it is easily verified that there must be a cyclic permutation $\langle a, b, c, d, a \rangle$ of $\pi$ or $\pi^{-1}$ such that one of the four digraphs in Fig. 3 matches $\rho$.

Now suppose the relation $\rho$ is the strict partial order $\nearrow$ introduced at the beginning of this section. Then digraph 3 in Fig. 3 cannot match $\rho = \nearrow$, since it is a directed cycle. We claim that digraph 2 also cannot match $\nearrow$. For suppose digraph 2 does match $\nearrow$. Let $e$ be the point $c + (c - d)$. Then $c \ l_1 e$, so $c \not\nearrow e$ (since $\pi$ is admissible) and hence either $c \nearrow e$ or $e \nearrow c$. The transitivity of $\nearrow$ implies $a \nearrow e$ in the former case (since $a \nearrow b$, $b \nearrow c$, and $c \nearrow e$), and implies $e \nearrow d$ in the latter case (since $e \nearrow c$ and $c \nearrow d$). But $a \nearrow e$ would imply $a \not\nearrow e$, while $e \nearrow d$ would imply $e \not\nearrow d$, and each of these contradicts the admissibility of $\pi$ because neither $a$ nor $d$ is $l_\infty$-adjacent to $e$. We have now proved:

**Lemma 3.3.** If $\pi$ is any unit square path in $\mathbb{Z}^n$, then there is a cyclic permutation $\langle a, b, c, d, a \rangle$ of $\pi$ or $\pi^{-1}$ such that either digraph 1 or digraph 4 in Fig. 3 matches $\nearrow$. 

![Fig. 3. If $\pi$ is any unit square path in $\mathbb{Z}^n$, and $\rho$ is any binary relation on $\mathbb{Z}^n$ such that $x \ l_1 y \Rightarrow x \rho y$, then there is a cyclic permutation $\langle a, b, c, d, a \rangle$ of $\pi$ or $\pi^{-1}$ such that one of these four digraphs matches $\rho$.](image-url)
Lemma 3.4. Let $\langle p, q \rangle$ be any $l_1$-path of length 1, then $p \equiv l_1 q$ and so $p \not\equiv q$, whence $p \not\equiv q$ or $q \not\equiv p$; in this case we define

$$h_\nearrow(\langle p, q \rangle) = +1 \quad \text{if} \quad p \not\equiv q,$$

$$h_\nearrow(\langle p, q \rangle) = -1 \quad \text{if} \quad q \not\equiv p.$$ 

More generally, for any $l_1$-path $\pi = \langle p_0, p_1, \ldots, p_m \rangle$, we define

$$h_\nearrow(\pi) = h_\nearrow(\langle p_0, p_1 \rangle) + h_\nearrow(\langle p_1, p_2 \rangle) + \cdots + h_\nearrow(\langle p_{m-1}, p_m \rangle).$$

(If the length of $\pi$ is 0 then $h_\nearrow(\pi) = 0$.) For example, if digraph 1 in Fig. 3 matches / then $h_\nearrow(\langle a, b, c, d \rangle) = -1 + 1 + 1 = 1$. Easy consequences of this definition are:

**Lemma 3.4.** Let $\pi$ be an arbitrary $l_1$-path in $\mathbb{Z}^n$. Then

1. $h_\nearrow(\pi^{-1}) = -h_\nearrow(\pi)$.
2. If $\pi = \pi_1 \cdot \pi_2 \cdot \ldots \cdot \pi_k$, then $h_\nearrow(\pi) = h_\nearrow(\pi_1) + h_\nearrow(\pi_2) + \cdots + h_\nearrow(\pi_k)$.
3. If $\pi$ is a closed $l_1$-path and $\pi^*$ is any cyclic permutation of $\pi$, then $h_\nearrow(\pi^*) = h_\nearrow(\pi)$.
4. $|h_\nearrow(\pi)|$ cannot exceed the length of $\pi$.

We confidently leave the proofs of these assertions to the reader. From Lemmas 3.3 and 3.4 we deduce:

**Lemma 3.5.** Let $\pi_1$ and $\pi_2$ be $l_1$-paths in $\mathbb{Z}^n$ with the same initial and the same final points, such that the sum of the lengths of $\pi_1$ and $\pi_2$ is at most 4. Then $h_\nearrow(\pi_1) = h_\nearrow(\pi_2)$.

**Proof.** Let $\pi = \pi_1 \cdot \pi_2^{-1}$. By Lemma 3.4, $h_\nearrow(\pi_1) - h_\nearrow(\pi_2) = h_\nearrow(\pi_1) + h_\nearrow(\pi_2^{-1}) = h_\nearrow(\pi_1 \cdot \pi_2^{-1}) = h_\nearrow(\pi)$. Thus it is enough to show that $h_\nearrow(\pi) = 0$.

$\pi$ is a closed $l_1$-path. There are no closed $l_1$-paths of length 1 or 3 (or of any odd length), so the length of $\pi$ is 2 or 4. If $\pi$ is of length 2 then $\pi = \langle p, q, p \rangle$ for some $p$ and $q$, and so $h_\nearrow(\pi) = 0$ as required. Thus we may assume $\pi$ is of length 4. Then $\pi = \langle p, q, r, s, p \rangle$ for some points $p, q, r,$ and $s$. Since consecutive points in $\pi$ are $l_1$-adjacent and hence distinct, either (i) $p, q, r, s$ are distinct, or (ii) $p = r$, or (iii) $q = s$. Evidently, $h_\nearrow(\pi) = 0$ in cases (ii) and (iii), so assume (i) is true. Then $\pi$ is a unit square path and by Lemma 3.3 there is a cyclic permutation $\langle a, b, c, d, a \rangle$ of $\pi$ or $\pi^{-1}$ such that digraph 1 or digraph 4 in Fig. 3 matches /'. Since digraph 1 or digraph 4 in Fig. 3 matches /', $h_\nearrow(\langle a, b, c, d, a \rangle) = 0$. As $\langle a, b, c, d, a \rangle$ is a cyclic permutation of $\pi$ or $\pi^{-1}$, and $h_\nearrow(\langle a, b, c, d, a \rangle) = 0$, Lemma 3.4 implies $h_\nearrow(\pi) = 0$, as required. \(\square\)

Say that two $l_1$-paths $\pi$ and $\pi'$ are 4-contiguous if there exist $l_1$-paths $\pi^*, \pi**, \pi_1$ and $\pi_2$ such that $\pi = \pi^* \cdot \pi_1 \cdot \pi^*$, $\pi' = \pi^* \cdot \pi_2 \cdot \pi^*$, and the sum of the lengths of $\pi_1$ and $\pi_2$ is at most 4. In this case Lemma 3.4 tells us that $h_\nearrow(\pi) = h_\nearrow(\pi^*) + h_\nearrow(\pi_1) + h_\nearrow(\pi^*)$ and $h_\nearrow(\pi') = h_\nearrow(\pi^*) + h_\nearrow(\pi_2) + h_\nearrow(\pi^*)$. So Lemma 3.5 implies:
Lemma 3.6. If $\pi$ and $\pi'$ are 4-contiguous $l_1$-paths in $\mathbb{Z}^n$, then $h_{\pi'}(\pi) = h_{\pi'}(\pi')$.

Note that in the above definition of 4-contiguity each of the $l_1$-paths $\pi^*, \pi^{**}, \pi_1$ and $\pi_2$ might be of length 0.

It is quite well known that $\mathbb{Z}^n$ has the following “$l_1$ simple connectedness” property (e.g., [3, Theorem 6.3.5]): The transitive closure of the 4-contiguity relation relates each closed $l_1$-path in $\mathbb{Z}^n$ to an $l_1$-path of length 0. This fact and Lemma 3.6 imply:

Proposition 3.7. $h_{\pi'}(\pi) = 0$ for all closed $l_1$-paths $\pi$ in $\mathbb{Z}^n$.

Corollary 3.8. Let $\pi$ and $\pi'$ be $l_1$-paths in $\mathbb{Z}^n$ that have the same initial and the same final points. Then $h_{\pi'}(\pi) = h_{\pi'}(\pi')$.

Proof. $h_{\pi'}(\pi') - h_{\pi'}(\pi) = h_{\pi'}(\pi') + h_{\pi'}(\pi^{-1}) = h_{\pi'}(\pi') \cdot \pi^{-1} = 0$ by Proposition 3.7. $\square$

For $p \in \mathbb{Z}^n$, we define the function $h_{\cdot} : \mathbb{Z}^n \to \mathbb{Z}$ by $h_{\cdot}(p) = h_{\cdot}(\pi)$, where $\pi$ is an arbitrary $l_1$-path in $\mathbb{Z}^n$ from $O = (0, 0, \ldots, 0)$ to $p$. This function is well defined, by Corollary 3.8. Note that $h_{\cdot}(O) = 0$.

3.3. Properties of $h_{\cdot} : \mathbb{Z}^n \to \mathbb{Z}$

Proposition 3.9. Let $x$ and $y$ be $l_1$-adjacent points in $\mathbb{Z}^n$. Then

$$h_{\cdot}(y) = h_{\cdot}(x) + 1 \quad \text{if} \quad x \not\sim y,$$

$$h_{\cdot}(y) = h_{\cdot}(x) - 1 \quad \text{if} \quad y \not\sim x.$$

Proof. Let $\pi$ be an $l_1$-path in $\mathbb{Z}^n$ from $O$ to $x$. Then $h_{\cdot}(x) = h_{\cdot}(\pi)$ and $h_{\cdot}(y) = h_{\cdot}(\pi \cdot \langle x, y \rangle)$, so $h_{\cdot}(y) = h_{\cdot}(\pi) + h_{\cdot}(\langle x, y \rangle) = h_{\cdot}(x) + 1$ or $h_{\cdot}(x) - 1$ according to whether $x \not\sim y$ or $y \not\sim x$. $\square$

Proposition 3.10. Let $x, y \in \mathbb{Z}^n$. Then

1. $h_{\cdot}(y) - h_{\cdot}(x) \equiv \|y - x\|_1$ (mod 2).
2. $|h_{\cdot}(y) - h_{\cdot}(x)| \leq \|y - x\|_1$.
3. If $h_{\cdot}(y) - h_{\cdot}(x) \equiv \|y - x\|_1 \not\equiv 0$, then $x \not\sim y$.
4. If $|h_{\cdot}(y) - h_{\cdot}(x)| \equiv \|y - x\|_1 \not\equiv 0$, then $x \not\sim y$.

Proof. Let $m = \|y - x\|_1$, and let $\langle x = p_0, p_1, \ldots, p_m = y \rangle$ be any $l_1$-path of length $m$ from $x$ to $y$. Then $h_{\cdot}(y) - h_{\cdot}(x) = \sum_{i=0}^{m-1} (h_{\cdot}(p_{i+1}) - h_{\cdot}(p_i))$.

By Proposition 3.9, each of the $m$ terms in this sum is $\equiv 1$ (mod 2), so the sum is $\equiv m$ (mod 2), proving 1. Moreover, the absolute value of each of the $m$ terms of the sum is 1, so the absolute value of the sum is $\leq m$, proving 2. Finally, if $h_{\cdot}(y) - h_{\cdot}(x) = \|y - x\|_1 = m > 0$ then (since the sum is $m$) each of the $m$ terms $h_{\cdot}(p_{i+1}) - h_{\cdot}(p_i)$ of the sum is 1. In this case $p_i \not\sim p_{i+1}$ for $0 \leq i < m$ (by Proposition 3.9) and so $x \not\sim y$ (by transitivity of $\not\sim$), proving 3. Since $x \not\sim y \Leftrightarrow x \not\sim y$ or $y \not\sim x$, 3 implies 4. $\square$
For $1 \leq i \leq n$, we write $v_i$ for the unit vector in the positive direction of the $i$th coordinate axis.

**Lemma 3.11.** Let $x \in \mathbb{Z}^n$ and let $y = x + 2v_i$ for some $i$. Then $h_\leftarrow(y) = h_\leftarrow(x)$.

**Proof.** Since $\|y - x\|_1 = 2$, assertions 1 and 2 of Proposition 3.10 imply $|h_\leftarrow(y) - h_\leftarrow(x)| = 0$ or 2. But $y$ is not $l_\infty$-adjacent to $x$, and hence $y$ is not $\pi$-adjacent to $x$ (since $\pi$ is admissible). So (by assertion 4 of Proposition 3.10) $|h_\leftarrow(y) - h_\leftarrow(x)| \neq \|y - x\|_1 = 2$ and so $h_\leftarrow(y) - h_\leftarrow(x) = 0$. □

**Corollary 3.12.** For all $x \in \mathbb{Z}^n$, $h_\leftarrow(x) = h_\leftarrow(x \mod 2)$.

Thus the function $h_\leftarrow : \mathbb{Z}^n \to \mathbb{Z}$ is determined by its restriction to the $2^n$ points in the unit $n$-cell $\{0, 1\}^n$.

**Proposition 3.13.** Let $x$, $p$ and $q$ be points in $\mathbb{Z}^n$ such that $p \nearrow q$. Then the vectors $p - x$ and $q - x$ are orthogonal to each other.

**Proof.** Suppose $p - x$ is not orthogonal to $q - x$. We claim that $x$ has an $l_1$-neighbor $y$ in $\mathbb{Z}^n$ such that neither $p$ nor $q$ is $l_\infty$-adjacent to $y$.

Indeed, $(p - x) \cdot (q - x) \neq 0$ and so there is some $j$ such that the $j$th components of $p - x$ and of $q - x$ are both nonzero. Now the $j$th components of $p - x$ and of $q - x$ must be 1 or $-1$, since $p \nearrow q \Rightarrow p \perp l_\infty q$; and they cannot have opposite signs since $p \nearrow q \Rightarrow p \perp l_\infty q$. Let $y = x - v_j$ or $x + v_j$ according to whether the $j$th components of $p - x$ and of $q - x$ are both equal to 1 or both equal to $-1$. Then the $j$th coordinate of $y$ differs from that coordinate of $p$ and of $q$ by 2, which justifies the claim.

Now $x \perp l_1 y \Rightarrow x \nearrow y$ or $y \nearrow x$, since $\pi$ is admissible. But $x \nearrow y$ would imply that $p \nearrow y$ (since $p \nearrow x$) and hence that $p \not\pi y$. Similarly, $y \nearrow x$ would imply that $y \nearrow q$ (since $x \nearrow q$) and hence that $y \not\pi q$. Both are impossible since $\pi$ is admissible and neither $p$ nor $q$ is $l_\infty$-adjacent to $y$. These contradictions prove the proposition. □

### 3.4. Consequences of $l_1$-connectedness

So far in this section we have not assumed that the adjacency $\pi$ is $l_1$-connected. In this section we will make that assumption: $\pi$ will denote an arbitrary $l_1$-connected admissible topological adjacency on $\mathbb{Z}^n$, and $\nearrow$ will (as before) denote a transitive orientation of $\pi$.

**Proposition 3.14.** Let $x \nearrow y$. Then there is an $l_1$-path from $x$ to $y$ such that $x \nearrow q$ for each point $q$ of the $l_1$-path other than the point $x$. 
Proof. As \( x \) is \( l_1 \)-connected and both \( x \) and \( y \) lie in the \( x \)-neighborhood of \( x \) (since \( x \not= y \)), there is an \( l_1 \)-path from \( x \) to \( y \) in the \( x \)-neighborhood of \( x \). Let \( (x = q_0, q_1, \ldots, q_k = y) \) be a shortest such \( l_1 \)-path. Then \( q_i \not= x \) for \( 1 \leq i \leq k \) (otherwise there would be a shorter \( l_1 \)-path from \( x \) to \( y \)). For \( 1 \leq i \leq k \), \( x \not= q_i \) and so \( x \not/ q_i \) or \( q_i \not/ x \). To prove the proposition, we now show that \( x \not/ q_i \) for \( 1 \leq i \leq k \).

We first claim that, for \( 1 \leq i \leq k \), the vectors \( q_{i-1} - x \) and \( q_i - x \) are not orthogonal to each other. Indeed, if \( v \) and \( w \) are vectors of \( \mathbb{Z}^n \) such that \( v \) has length 1, then \( w \cdot v \leq |w||v| = |w| \leq w \cdot w \). If \( w \) is nonzero and \( w \not= v \) then at least one of the two \( \leq \)'s here can be replaced by \( < \), and so \( w \cdot (w - v) > 0 \). The claim follows from this on putting \( w = q_{i-1} - x \) and \( v = q_i - q_{i-1} \).

For any \( i \) in the range \( 1 < i \leq k \), \( x \not/ q_i \) implies that \( q_{i-1} \not/ x \) is false (by Proposition 3.13 and the above claim), which in turn implies that \( x \not/ q_{i-1} \). So, since \( x \not/ q_k = y \), we have (by induction) \( x \not/ q_i \) for \( 1 \leq i \leq k \). \( \square \)

Proposition 3.15. For \( x, y \in \mathbb{Z}^n \),
\[
x \not/ y \iff h_\prec (y) - h_\prec (x) = \|y - x\|_1 \neq 0.
\]

Proof. \( \Leftarrow \) is assertion 3 of Proposition 3.10. We now prove \( \Rightarrow \).

Let \( x \not/ y \). Then \( \|y - x\|_1 \neq 0 \). There is an \( l_1 \)-path \( (x = q_0, q_1, \ldots, q_k = y) \) such that \( x \not/ q_i \) for \( 1 \leq i \leq k \) (by Proposition 3.14). We claim that \( h_\prec (q_i) - h_\prec (x) = \|q_i - x\|_1 \) for all \( i \). This case \( i = k \) of this claim is the \( \Rightarrow \) part of the proposition.

The claim is true for \( i = 0 \), because \( x = q_0 \). Suppose that for some \( j < k \) the claim is true for \( i = j \). To complete the proof, we will deduce that \( h_\prec (q_{j+1}) - h_\prec (q_j) = \|q_{j+1} - x\|_1 - \|q_j - x\|_1 \neq 0 \), so that the claim is also true for \( i = j + 1 \).

Either (a) \( h_\prec (q_{j+1}) - h_\prec (q_j) = -1 \) or (b) \( h_\prec (q_{j+1}) - h_\prec (q_j) = 1 \) (by Proposition 3.9).

Suppose (a) holds. Then \( q_{j+1} \not/ q_j \). As \( x \not/ q_{j+1} \), the vectors \( q_{j+1} - x \) and \( q_j - q_{j+1} \) are orthogonal (by Proposition 3.13), and so \( \|q_{j+1} - x\|_1 + \|q_j - q_{j+1}\|_1 = \|q_{j+1} - x\| + \|q_j - q_{j+1}\| \). Therefore, \( \|q_{j+1} - x\|_1 - \|q_j - x\|_1 = -\|q_j - q_{j+1}\| = -1 = h_\prec (q_{j+1}) - h_\prec (q_j) \), as required. Now suppose instead that (b) holds. Then \( x \not/ q_{j+1} \), so the vectors \( q_j - x \) and \( q_{j+1} - q_j \) are orthogonal. Hence, \( \|q_j - x\|_1 + \|q_{j+1} - q_j\|_1 = \|q_j - x\| + \|q_{j+1} - q_j\| \) and so \( \|q_{j+1} - x\|_1 - \|q_j - x\|_1 = \|q_{j+1} - q_j\|_1 = 1 = h_\prec (q_{j+1}) - h_\prec (q_j) \), as required. \( \square \)

Let \( \theta_\prec : \{0,1\}^n \to \mathbb{Z}^n \) denote the restriction of \( h_\prec \) to \( \{0,1\}^n \). Then \( \theta_\prec \) is an admissible function. We now state the principal result of Section 3:

Theorem 3.16. For \( x, y \in \mathbb{Z}^n \),
1. \( x \not/ y \iff \theta_\prec (y \mod 2) - \theta_\prec (x \mod 2) = \|y - x\|_1 \neq 0 \).
2. \( x \not= y \iff \theta_\prec (y \mod 2) - \theta_\prec (x \mod 2) \neq \|y - x\|_1 \neq 0 \).

Proof. By Corollary 3.12, \( h_\prec (p) = \theta_\prec (p \mod 2) \). So Proposition 3.15 implies assertion 1. Since \( x \not= y \iff x \not/ y \) or \( y \not/ x \), assertion 1 implies assertion 2. \( \square \)
Assertion 2 of this theorem shows that the admissible function \( \theta \) completely determines the admissible \( l_1 \)-connected topological adjacency \( \varepsilon \). Note that assertion 2 remains true if \( \theta \) is changed to its complement \( \overline{\theta} = -\theta \).

4. The main theorems

The two main theorems of this paper are:

**Theorem 4.1.** Let \( \varepsilon \) be an admissible \( l_1 \)-connected topological adjacency on \( \mathbb{Z}^n \). Then there are just two admissible functions \( \theta : \{0,1\}^n \to \mathbb{Z} \) such that, for \( x, y \in \mathbb{Z}^n \),

\[
x \varepsilon y \iff |\theta(y \mod 2) - \theta(x \mod 2)| \neq 0.
\]

These two functions constitute a complementary pair.

**Theorem 4.2.** Let \( \theta : \{0,1\}^n \to \mathbb{Z} \) be an admissible function. Then the relation \( \varepsilon \) on \( \mathbb{Z}^n \) defined by

\[
x \varepsilon y \iff |\theta(y \mod 2) - \theta(x \mod 2)| \neq 0
\]

is an admissible \( l_1 \)-connected topological adjacency.

Theorem 4.1 gives us a mapping of admissible \( l_1 \)-connected topological adjacencies on \( \mathbb{Z}^n \) to complementary pairs of admissible functions. Theorem 4.2 shows that this mapping has an inverse, and is therefore a bijection.

4.1. Proof of Theorem 4.1

By assertion 2 of Theorem 3.16, condition (5) is satisfied by a complementary pair of admissible functions—namely \( \theta \) and \( \overline{\theta} \), where \( \overline{\theta} \) is a transitive orientation of \( \varepsilon \). It remains only to show that any admissible function \( \theta \) which satisfies (5) must be equal to one of the two functions of this existing complementary pair. We will deduce this from:

**Lemma 4.3.** Let \( \theta_i : \{0,1\}^n \to \mathbb{Z} \) be admissible functions such that (5) holds for \( \theta = \theta_i \) (i = 1, 2). Let \( \langle p, q, r \rangle \) be an \( l_1 \)-path in \( \{0,1\}^n \) such that \( \theta_1(p) = \theta_2(p) \) and \( \theta_1(q) = \theta_2(q) \). Then \( \theta_1(r) = \theta_2(r) \).

**Proof.** The result is trivial if \( r = p \), so we may assume \( r \neq p \). For \( i = 1 \) or 2, the admissibility of \( \theta_i \) implies that one of the following is true:

(a) \( \theta_i(r) - \theta_i(q) = \theta_i(q) - \theta_i(p) \)
(b) \( \theta_i(r) - \theta_i(q) = - (\theta_i(q) - \theta_i(p)) \)

In case (a), \( |\theta_i(r) - \theta_i(p)| = 2|\theta_i(q) - \theta_i(p)| = 2 = \|r - p\|_1 \), so (5) implies \( p \varepsilon r \). In case (b), \( \theta_i(r) - \theta_i(p) = 0 \), so (5) implies that \( p \varepsilon r \) is false.
Thus if \( p \preceq r \) is true then (a) holds for both \( i = 1 \) and \( i = 2 \), while if \( p \preceq r \) is false then (b) holds for both \( i = 1 \) and \( i = 2 \). In (a) and (b), the 2nd, 3rd and 4th terms have the same values for \( i = 1 \) as for \( i = 2 \). So in both cases the 1st term, \( \theta_i(r) \), also has the same value for \( i = 1 \) as for \( i = 2 \). This proves the lemma. \( \square \)

Since \( \{0, 1\}^n \) is \( l_1 \)-connected, it follows from this lemma (by induction) that if an admissible function that satisfies (5) has the same value as another such function at each of two \( l_1 \)-adjacent points, then the two admissible functions are equal.

Let \( \theta^* \) be an admissible function that satisfies (5), and let \( z \) be an \( l_1 \)-neighbor of the origin \( O \). Every admissible function has value 0 at \( O \), and has value 1 or \(-1 \) at \( z \) (by Proposition 3.9). As mentioned above, we know there is a complementary pair of admissible functions that satisfy (5). Since one function of this pair has value 1 at \( z \) while the other has value \(-1 \), \( \theta^* \) has the same value at both \( O \) and \( z \) as one of the functions of the pair. As we observed in the previous paragraph, this implies \( \theta^* \) is equal to that function. This completes the proof of Theorem 4.1.

### 4.2. Proof of Theorem 4.2

Evidently \( z \) is irreflexive and symmetric, and \( x \parallel_l y \Rightarrow x \preceq y \) since \( \theta \) is admissible. Now let \( x \) and \( y \) be distinct points in \( \mathbb{Z}^n \) that are not \( l_\infty \)-adjacent. Then there is a coordinate of \( x \) that differs from the same coordinate of \( y \) by at least 2, and so \( x \sim_l y \) if and only if \( \parallel x \parallel_l > \parallel (y \mod 2) - (x \mod 2) \parallel_1 \). But we know that \( \parallel (y \mod 2) - (x \mod 2) \parallel_1 \geq \parallel (y \mod 2) - \theta(y \mod 2) - \theta(x \mod 2) \parallel_1 \) (this is (4) in Section 3.1). So \( x \sim_l y \) if and only if \( \parallel y - x \parallel_1 > \parallel (y \mod 2) - \theta(y \mod 2) - \theta(x \mod 2) \parallel_1 \), whence \( x \preceq y \) is false. This shows that \( x \parallel_l y \) if and only if \( x \parallel_l \infty \).

We have now verified that \( \preceq \) is an admissible adjacency. It remains to show that \( \parallel \) is \( l_1 \)-connected and topological. Let \( x \) be any point in \( \mathbb{Z}^n \) and let \( y \) be any other point in the \( \infty \)-neighborhood of \( x \). Since \( x \parallel_l y \), it follows from Corollary 3.2 that there is a \( \theta \)-monotonic \( l_1 \)-path from \( x \) to \( y \). Let \( \pi = \langle x = q_0, q_1, \ldots, q_m = y \rangle \) be such an \( l_1 \)-path. As \( \pi \) is \( \theta \)-monotonic, \( q_j \neq x \) for \( 1 \leq j \leq m \). For \( 1 \leq j \leq m \), there is a \( \theta \)-monotonic \( l_1 \)-path in \( \mathbb{Z}^n \) from \( x \) to \( q_j \), namely \( \langle x = q_0, q_1, \ldots, q_j \rangle \), and so it follows from Corollary 3.2 that \( x \parallel_l q_j \). Thus, \( \langle x = q_0, q_1, \ldots, q_m = y \rangle \) is an \( l_1 \)-path from \( x \) to \( y \) within the \( \infty \)-neighborhood of \( x \). This shows that \( \parallel \) is an \( l_1 \)-connected adjacency.

Let \( \not\parallel \) be the binary relation on \( \mathbb{Z}^n \) such that \( x \not\parallel y \) if and only if \( x \neq y \) and there exists a \( \theta \)-increasing \( l_1 \)-path in \( \mathbb{Z}^n \) from \( x \) to \( y \). Plainly, \( \not\parallel \) is a strict partial order. Moreover, it follows from Proposition 3.1 that, for \( x, y \in \mathbb{Z}^n \), \( x \parallel_l y \iff x \not\parallel y \) or \( y \not\parallel x \). Hence \( \parallel \) is topological, by Theorem 2.1. This completes the proof of Theorem 4.2.

### 5. Isomorphisms and isometries

#### 5.1. Isomorphism of adjacencies

Two adjacencies \( \preceq_1 \) and \( \preceq_2 \) on \( \mathbb{Z}^n \) are said to be isomorphic if there is a bijection \( \sigma : \mathbb{Z}^n \rightarrow \mathbb{Z}^n \) such that \( x \parallel_1 y \iff \sigma(x) \parallel_2 \sigma(y) \); such a bijection \( \sigma \) is called an isomorphism of \( \preceq_1 \) to \( \preceq_2 \).
We now show that if $x_1$ and $x_2$ are admissible $l_1$-connected topological adjacencies then any isomorphism of $x_1$ to $x_2$ must map $l_1$-neighbors to $l_1$-neighbors. In fact, this implies that any such isomorphism must be an isometry of $\mathbb{Z}^n$ onto itself.

**Theorem 5.1.** Let $x_1$ and $x_2$ be isomorphic admissible $l_1$-connected topological adjacencies on $\mathbb{Z}^n$, let $\sigma$ be an isomorphism of $x_1$ to $x_2$, and let $x$ and $y$ be $l_1$-adjacent points in $\mathbb{Z}^n$. Then $\sigma(x) \parallel l_1 \sigma(y)$.

**Proof.** Let $\frac{\rightarrow}{2}$ be a transitive orientation of $x_2$, and let $\frac{\rightarrow}{1}$ be the strict partial order on $\mathbb{Z}^n$ defined by $p \frac{\rightarrow}{1} q \iff \sigma(p) \frac{\rightarrow}{2} \sigma(q)$. Then $\frac{\rightarrow}{1}$ is a transitive orientation of $x_1$.

Since $x \perp l_1 y$, we have that $x$ $\perp l_1 y$ and hence either $x \frac{\rightarrow}{1} y$ or $y \frac{\rightarrow}{1} x$. We may assume, without loss of generality, that $x \frac{\rightarrow}{1} y$. Then $\sigma(x) \frac{\rightarrow}{2} \sigma(y)$.

Let $m = ||\sigma(y) - \sigma(x)||_1$, and let $\pi = (\sigma(x) = q_0, q_1, \ldots, q_m = \sigma(y))$ be an $l_1$-path of length $m$ from $\sigma(x)$ to $\sigma(y)$. As $\sigma(x) \frac{\rightarrow}{2} \sigma(y)$, it follows from Proposition 3.15 that $h_{\frac{\rightarrow}{2}}(\sigma(y)) - h_{\frac{\rightarrow}{1}}(\sigma(x)) = m$.

Thus $\sum_{i=1}^{m} (h_{\frac{\rightarrow}{1}}(q_i) - h_{\frac{\rightarrow}{2}}(q_{i-1})) = m$ and so, since each of the $m$ terms in this sum has value $1$ or $-1$ (by Proposition 3.9), $h_{\frac{\rightarrow}{2}}(q_i) - h_{\frac{\rightarrow}{1}}(q_{i-1}) = 1$ for $1 \leq i \leq m$. Hence, for $1 \leq i \leq m$, $q_{i-1} \frac{\rightarrow}{2} q_i$, whence $\sigma^{-1}(q_{i-1}) \frac{\rightarrow}{1} \sigma^{-1}(q_i)$, whence $h_{\frac{\rightarrow}{1}}(\sigma^{-1}(q_{i-1})) - h_{\frac{\rightarrow}{2}}(\sigma^{-1}(q_i)) \geq 1$ (by Proposition 3.15), whence

$$h_{\frac{\rightarrow}{1}}(y) - h_{\frac{\rightarrow}{2}}(x) = h_{\frac{\rightarrow}{1}}(\sigma^{-1}(q_m)) - h_{\frac{\rightarrow}{2}}(\sigma^{-1}(q_0)) \geq m.$$

However, $x \frac{\rightarrow}{1} y$, so $h_{\frac{\rightarrow}{1}}(y) - h_{\frac{\rightarrow}{2}}(x) = 1$ by Proposition 3.9. Therefore $m = 1$ and so $\sigma(x) \perp l_1 \sigma(y)$. \Box

In view of this theorem, we are interested in the nature of bijections of $\mathbb{Z}^n$ onto itself that map $l_1$-neighbors to $l_1$-neighbors. Later on we will also need to consider bijections of $\{0, 1\}^n$ onto itself that map $l_1$-neighbors to $l_1$-neighbors. It turns out that all such bijections can be expressed as a composition of rotations, reflections and translations of the following kinds:

1. For any permutation $\zeta$ of $\{1, 2, \ldots, n\}$, let $\mu_\zeta$ denote the map with domain $\mathbb{Z}^n$ or $\{0, 1\}^n$ such that $\mu_\zeta(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$, where $y_i = x_{\zeta(i)}$ for $1 \leq i \leq n$.

2. For any subset $S$ of $\{1, 2, \ldots, n\}$, let $\mu_S$ denote the map with domain $\mathbb{Z}^n$ or $\{0, 1\}^n$ such that $\mu_S(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$, where $y_i = x_i$ if $i \notin S$ and $y_i = 1 - x_i$ if $i \in S$.

3. For any vector $v$ of $\mathbb{Z}^n$, let $\mu_v$ be the map with domain $\mathbb{Z}^n$ such that $\mu_v(x) = x + v$.

**Fact 5.2.** If $\psi: \{0, 1\}^n \rightarrow \{0, 1\}^n$ is a bijection that maps $l_1$-neighbors to $l_1$-neighbors, then there is a unique permutation $\zeta$ of $\{1, 2, \ldots, n\}$ and a unique subset $S$ of $\{1, 2, \ldots, n\}$ such that $\mu_S \circ \mu_\zeta = \psi$.

**Fact 5.3.** If $\sigma: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ is a bijection that maps $l_1$-neighbors to $l_1$-neighbors, then there is a unique permutation $\zeta$ of $\{1, 2, \ldots, n\}$, a unique subset $S$ of $\{1, 2, \ldots, n\}$, and a unique vector $v$ of $\mathbb{Z}^n$ such that $\mu_v \circ \mu_S \circ \mu_\zeta = \sigma$. 
We leave the proofs of these two facts to the reader. These facts imply that every bijection of \( \{0,1\}^n \) or \( \mathbb{Z}^n \) onto itself which maps \( l_1 \)-neighbors to \( l_1 \)-neighbors is an isometry (with respect to the usual Euclidean metric). This justifies our earlier assertion that every isomorphism of one admissible \( l_1 \)-connected topological adjacency on \( \mathbb{Z}^n \) to another is an isometry.

Conversely, since any isometry of \( \mathbb{Z}^n \) onto itself is a bijection that maps \( l_1 \)-neighbors, and only \( l_1 \)-neighbors, to \( l_1 \)-neighbors, and maps \( l_\infty \)-neighbors to \( l_\infty \)-neighbors (e.g., by Fact 5.3), it is easy to see that every isometry of \( \mathbb{Z}^n \) onto itself is an isomorphism of any admissible \( l_1 \)-connected topological adjacency on \( \mathbb{Z}^n \) to another such adjacency.

5.2. Isomorphism of functions on subsets of \( \{0,1\}^n \)

If \( f \) is any function then we will write \( \mathcal{D}_f \) for the domain of \( f \).

We now define another concept of isomorphism, which will be needed in the next section. Let \( f_1 \) and \( f_2 \) be functions such that \( \mathcal{D}_{f_1}, \mathcal{D}_{f_2} \subseteq \{0,1\}^n \). We say that \( f_1 \) and \( f_2 \) are isomorphic if there is an isometry \( \psi \) of \( \{0,1\}^n \) onto itself such that \( \psi[\mathcal{D}_{f_1}] = \mathcal{D}_{f_2} \) and \( f_1 = f_2 \circ \psi \) on \( \mathcal{D}_{f_1} \). Such an isometry \( \psi \) will be called an isomorphism of \( f_1 \) to \( f_2 \).

6. Normalized-admissible functions

A drawback of the concept of an admissible function is that a function \( \phi' \) which is isomorphic to an admissible function \( \phi \) need not itself be an admissible function: An isomorphism of \( \phi' \) to \( \phi \) may be an isometry \( \psi \) that maps the origin \( O \) to a point at which \( \phi \) is non-zero, in which case \( \phi'(O) = \phi(\psi(O)) \neq 0 \) and so \( \phi' \) is not admissible.

We now introduce a closely related concept that does not have this deficiency.

Let \( N \) denote the set of non-negative integers. A normalized-admissible function is a function \( \phi: \{0,1\}^n \to N \) that has the following two properties:

(A) Whenever \( x, y \), \( |\phi(y) - \phi(x)| = 1 \).

(B) \( \phi(x) = 0 \) for at least one point \( x \in \{0,1\}^n \).

For any normalized-admissible function \( \phi: \{0,1\}^n \to N \), the complement of \( \phi \), denoted by \( \check{\phi} \), is defined by \( \check{\phi}(x) = \max_{y \in \{0,1\}^n} \phi(y) - \phi(x) \) for all \( x \in \{0,1\}^n \). Evidently, \( \check{\phi} \) is a normalized-admissible function on \( \{0,1\}^n \). Note that \( \phi(x) - \check{\phi}(y) = \phi(y) - \phi(x) \) for any \( x, y \in \{0,1\}^n \). The unordered pair \( \{\phi, \check{\phi}\} \) is called a complementary pair.

6.1. Relationship to admissible functions

If \( \phi: \{0,1\}^n \to N \) is any normalized-admissible function then we define \( \phi^-: \{0,1\}^n \to \mathbb{Z} \) by \( \phi^-(x) = \phi(x) - \phi(O) \). If \( \theta: \{0,1\}^n \to \mathbb{Z} \) is any admissible function then we define \( \theta^+: \{0,1\}^n \to N \) by \( \theta^+(x) = \theta(x) - \min_{y \in \{0,1\}^n} \theta(y) \). (Note that \( \min_{y \in \{0,1\}^n} \theta(y) \leq \theta(O) = 0 \).) It is readily confirmed that the mapping \( \phi \mapsto \phi^- \) maps each normalized-admissible function to an admissible function, while the mapping \( \theta \mapsto \theta^+ \) maps each admissible function to a normalized-admissible function, and that these two mappings
are inverses of each other. Thus \( \theta \mapsto \theta^+ \) and \( \phi \mapsto \phi^- \) are mutually inverse bijections of the admissible functions on \( \{0,1\}^n \) onto the normalized-admissible functions on \( \{0,1\}^n \), and vice versa. These bijections induce bijections of the complementary pairs of admissible functions onto the complementary pairs of normalized-admissible functions, and vice versa (since \( \theta^+ = \overline{\theta^+} \) and \( \phi^- = \overline{\phi^-} \) for any admissible function \( \theta \) and any normalized-admissible function \( \phi \)).

For any admissible function \( \theta \) and any \( x, y \in \{0,1\}^n \), the normalized-admissible function \( \theta^+ \) satisfies \( \theta^+(y) - \theta^+(x) = \theta(y) - \theta(x) \). Conversely, for any normalized-admissible function \( \phi \) the admissible function \( \phi^- \) satisfies \( \phi^-(y) - \phi^-(x) = \phi(y) - \phi(x) \). From these observations and Theorems 4.1 and 4.2 we deduce the following results:

**Theorem 6.1.** If \( \alpha \) is an \( l_1 \)-connected topological adjacency on \( \mathbb{Z}^n \), then there are just two normalized-admissible functions \( \phi : \{0,1\}^n \to \mathbb{N} \) such that
\[
x \alpha y \iff |\phi(y \mod 2) - \phi(x \mod 2)| = \|y - x\|_1 \neq 0
\]
for all \( x, y \in \mathbb{Z}^n \). These two functions constitute a complementary pair.

**Theorem 6.2.** If \( \phi : \{0,1\}^n \to \mathbb{N} \) is a normalized-admissible function, then the relation \( \alpha \) on \( \mathbb{Z}^n \) defined by
\[
x \alpha y \iff |\phi(y \mod 2) - \phi(x \mod 2)| = \|y - x\|_1 \neq 0
\]
is an admissible \( l_1 \)-connected topological adjacency.

6.2. Isomorphism of complementary pairs

An isomorphism of the complementary pair of normalized-admissible functions \( \{\phi_1, \overline{\phi_1}\} \) to the complementary pair \( \{\phi_2, \overline{\phi_2}\} \) of such functions is an isometry of \( \{0,1\}^n \) onto \( \{0,1\}^n \) that is either an isomorphism of \( \phi_1 \) to \( \phi_2 \) (and hence also of \( \overline{\phi_1} \) to \( \overline{\phi_2} \)) or an isomorphism of \( \overline{\phi_1} \) to \( \phi_2 \) (and hence also of \( \phi_1 \) to \( \overline{\phi_2} \)). Naturally, if such an isometry exists then we say that the two complementary pairs are isomorphic.

Let \( \sigma \) be any isometry of \( \mathbb{Z}^n \) onto itself. Then we define \( \sigma_{\{0,1\}} : \{0,1\}^n \to \{0,1\}^n \) by \( \sigma_{\{0,1\}}(x) = \sigma(x) \mod 2 \). If \( x \) and \( y \) are distinct points in \( \{0,1\}^n \), then \( x \) and \( y \) differ by 1 in some coordinate, so \( \sigma(x) \) and \( \sigma(y) \) must also differ by exactly 1 in some coordinate (e.g., by Fact 5.3), whence \( \sigma(x) \mod 2 \neq \sigma(y) \mod 2 \). Thus \( \sigma_{\{0,1\}} : \{0,1\}^n \to \{0,1\}^n \) is 1–1 and must therefore be a bijection. Since \( \sigma \) is an isometry, it maps \( l_1 \)-neighbors to \( l_1 \)-neighbors. Hence, \( \sigma_{\{0,1\}} \) also maps \( l_1 \)-neighbors to \( l_1 \)-neighbors and is an isometry of \( \{0,1\}^n \) onto itself (e.g., by Fact 5.2).

**Proposition 6.3.** Let \( \sigma \) be an isometry of \( \mathbb{Z}^n \) onto itself. Then, for all \( q \in \mathbb{Z}^n \), \( \sigma(q) \mod 2 = \sigma_{\{0,1\}}(q \mod 2) \).

**Proof.** Let \( x \in \mathbb{Z}^n \) and let \( v \) be any unit vector of \( \mathbb{Z}^n \). As \( \sigma \) is an isometry of \( \mathbb{Z}^n \) onto itself, \( \sigma(x) \) is at a distance of 1 from each of \( \sigma(x + v) \) and \( \sigma(x - v) \), and \( \sigma(x + v) \) is at a
distance of 2 from $\sigma(x - v)$. It follows that there is some unit vector $w$ of $\mathbb{Z}^n$ such that $\sigma(x + v) = \sigma(x) + w$ and $\sigma(x - v) = \sigma(x) - w$. Thus $\sigma(x + v) \mod 2 = \sigma(x - v) \mod 2$.

As $x$ is an arbitrary point in $\mathbb{Z}^n$ and $v$ an arbitrary unit vector of $\mathbb{Z}^n$, it follows that $\sigma(q) \mod 2 = \sigma(q \mod 2) \mod 2$ for all $q \in \mathbb{Z}^n$. □

**Theorem 6.4.** Let $\phi_1, \phi_2 : \{0, 1\}^n \rightarrow \mathbb{N}$ be normalized-admissible functions and, for $i = 1$ and 2, let $\alpha_i$ be the admissible $l_1$-connected topological adjacency on $\mathbb{Z}^n$ such that $\alpha = \alpha_i$ satisfies (6) in Theorem 6.2 when $\phi = \phi_i$. Let $\sigma$ be an isometry of $\mathbb{Z}^n$ onto itself. Then $\sigma$ is an isomorphism of $\alpha_1$ to $\alpha_2$ if and only if $\sigma(01)$ is an isomorphism of $\{\phi_1, \overline{\phi}_1\}$ to $\{\phi_2, \overline{\phi}_2\}$.

**Proof.** By our definition of isomorphism of adjacencies, $\sigma$ is an isomorphism of $\alpha_1$ to $\alpha_2$ if and only if $x \alpha_1 y \Leftrightarrow \sigma(x) \alpha_2 \sigma(y)$. But, by (6), the latter is true if and only if

$$\{\phi_1(y \mod 2) - \phi_1(x \mod 2) = \|y - x\|_1 \neq 0 \} \Leftrightarrow \{\phi_2(\sigma(y) \mod 2) - \phi_2(\sigma(x) \mod 2) = \|\sigma(y) - \sigma(x)\|_1 \neq 0 \}.$$  

(7)

Since $\sigma$ is an isometry of $\mathbb{Z}^n$ onto itself, $\sigma$ and $\sigma^{-1}$ map $l_1$-paths to $l_1$-paths of the same length, and so $\|\sigma(y) - \sigma(x)\|_1 = \|y - x\|_1$. Hence it follows from Proposition 6.3 that (7) is true if and only if

$$\{\phi_1(y \mod 2) - \phi_1(x \mod 2) = \|y - x\|_1 \neq 0 \} \Leftrightarrow \{\phi_2(\sigma_{\{0,1\}}(y \mod 2)) - \phi_2(\sigma_{\{0,1\}}(x \mod 2)) = \|y - x\|_1 \neq 0 \}.$$  

(8)

Evidently, equivalence (8) holds if $\phi_2 \circ \sigma_{\{0,1\}}$ is equal to $\phi_1$ or to $\overline{\phi}_1$. Conversely, if equivalence (8) holds then both of the following equivalences hold:

$$x \alpha_1 y \Leftrightarrow \|\phi_1(y \mod 2) - \phi_1(x \mod 2)\|_1 = \|y - x\|_1 \neq 0,$$

$$x \alpha_1 y \Leftrightarrow \phi_2 \circ \sigma_{\{0,1\}}(y \mod 2) - \phi_2 \circ \sigma_{\{0,1\}}(x \mod 2) = \|y - x\|_1 \neq 0,$$

and so by Theorem 6.1 the normalized-admissible function $\phi_2 \circ \sigma_{\{0,1\}}$ must be equal to $\phi_1$ or to $\overline{\phi}_1$. □

**Corollary 6.5.** Under the hypotheses of Theorem 6.4, the adjacencies $\alpha_1$ and $\alpha_2$ are isomorphic if and only if the corresponding complementary pairs $\{\phi_1, \overline{\phi}_1\}$ and $\{\phi_2, \overline{\phi}_2\}$ are isomorphic.

**Proof.** The “only if” part is an immediate consequence of Theorem 6.4. For the “if” part, let $\psi : \{0, 1\}^n \rightarrow \{0, 1\}^n$ be an isomorphism of $\phi_1$ or $\overline{\phi}_1$ to $\phi_2$. Then there is an isometry $\sigma$ of $\mathbb{Z}^n$ onto itself whose restriction to $\{0, 1\}^n$ is equal to $\psi$ (e.g., by Fact 5.2). As $\sigma$ maps $\{0, 1\}^n$ onto itself, $\sigma_{\{0,1\}}(x) = \sigma(x) \mod 2 = \sigma(x) = \psi(x)$ for all $x \in \{0, 1\}^n$. Since $\sigma_{\{0,1\}} = \psi$ is an isomorphism of $\phi_1$ or $\overline{\phi}_1$ to $\phi_2$, it follows from Theorem 6.4 that $\sigma$ is an isomorphism of $\alpha_1$ to $\alpha_2$. □
It follows from this corollary that our bijection of the admissible \( l_1 \)-connected topological adjacencies on \( \mathbb{Z}^n \) onto the complementary pairs of normalized-admissible functions on \( \{0,1\}^n \) induces a bijection of the isomorphism classes of such adjacencies onto the isomorphism classes of such pairs.

7. \( l_1 \)-local minima and admissible labelings

7.1. Another bijection

We say that a point \( a \in \{0,1\}^n \) is an \( l_1 \)-local minimum of a normalized-admissible function \( \phi : \{0,1\}^n \to \mathbb{N} \) if for every \( l_1 \)-neighbor \( x \) of \( a \) in \( \{0,1\}^n \) we have \( \phi(x) > \phi(a) \) (equivalently, \( \phi(x) = \phi(a) + 1 \)).

We define an admissible labeling of \( \{0,1\}^n \) to be a non-negative-integer-valued mapping \( \phi \) such that \( \phi \) and its domain \( D_\phi \) satisfy the following four conditions:

1. \( D_\phi \) is a non-empty subset of \( \{0,1\}^n \).
2. For all \( a, b \in D_\phi \), \( \phi(a) - \phi(b) \equiv \|a - b\|_1 \mod 2 \).
3. For all distinct points \( a, b \in D_\phi \), \( |\phi(a) - \phi(b)| \leq \|a - b\|_1 - 2 \).
4. \( \min_{a \in D_\phi} \phi(a) = 0 \).

Note that condition 3 implies no two points in the domain \( D_\phi \) of \( \phi \) are \( l_1 \)-adjacent, and that conditions 3 and 4 imply \( \phi(x) \leq n - 2 \) for all \( x \in D_\phi \).

The following two theorems tell us that

\[
\phi(x) = \min_{a \in D_\phi} (\phi(a) + \|x - a\|_1)
\]

defines a bijection of the admissible labelings \( \phi \) of \( \{0,1\}^n \) onto the normalized-admissible functions \( \phi \) on \( \{0,1\}^n \).

**Theorem 7.1.** Let \( \phi : \{0,1\}^n \to \mathbb{N} \) be a normalized-admissible function. Then there is a unique admissible labeling \( \phi \) of \( \{0,1\}^n \) such that

\[
\phi(x) = \min_{a \in D_\phi} (\phi(a) + \|x - a\|_1)
\]

for all \( x \in \{0,1\}^n \). This \( \phi \) is the restriction of \( \phi \) to the set of all \( l_1 \)-local minima of \( \phi \).

**Theorem 7.2.** Let \( \phi \) be an admissible labeling of \( \{0,1\}^n \). Then the function \( \phi : \{0,1\}^n \to \mathbb{N} \) defined by

\[
\phi(x) = \min_{a \in D_\phi} (\phi(a) + \|x - a\|_1)
\]

is a normalized-admissible function.
7.2. Proof of Theorem 7.1

Let \( \varphi \) be the restriction of \( \phi \) to the set of all \( l_1 \)-local minima of \( \phi \). Then \( \varphi \) satisfies condition 1 in the definition of an admissible labeling of \( \{0,1\}^n \). We now show that \( \varphi \) also satisfies conditions 2, 3 and 4.

Since \( \phi \) is a normalized-admissible function, \( \phi(a_0) = 0 \) for some \( a_0 \in \{0,1\}^n \). Such a point \( a_0 \) is an \( l_1 \)-local minimum of \( \phi \) and is therefore in \( D_\varphi \); moreover, \( \varphi(a_0) = 0 \) as \( \varphi = \phi \) on \( D_\varphi \). So \( \varphi \) satisfies condition 4.

To see that \( \varphi \) satisfies condition 2, let \( a \) and \( b \) be any points in \( D_\varphi \) and consider a shortest \( l_1 \)-path in \( \{0,1\}^n \) from \( a \) to \( b \). The length of such an \( l_1 \)-path is \( \|a - b\|_1 \) and so, since \( \phi \) changes parity when one takes a step along an \( l_1 \)-path, \( \phi(a) - \phi(b) = \phi(a) - \phi(b) \) has the same parity as \( \|a - b\|_1 \), as required.

Now we verify condition 3. Let \( w \) and \( x \) be arbitrary points in \( \{0,1\}^n \). Then there is an \( l_1 \)-path of length \( \|x - w\|_1 \) from \( w \) to \( x \). Suppose we step along the \( l_1 \)-path from \( w \) to \( x \). Since \( \phi \) changes in value by \( \pm 1 \) at each step,

\[
|\phi(x) - \phi(w)| \leq \|x - w\|_1.
\]

If \( w, x \in D_\varphi \) and \( w \neq x \), then \( \phi \) changes by \( +1 \) at the first step, and by \( -1 \) at the last step (since \( w \) and \( x \) are \( l_1 \)-local minima of \( \phi \)), so the changes at these two steps cancel each other out and, as there are only \( \|x - w\|_1 - 2 \) other steps, \( |\phi(x) - \phi(w)| = |\phi(x) - \phi(w)| \leq \|x - w\|_1 - 2 \). So condition 3 holds. Thus \( \varphi \) is an admissible labeling of \( \{0,1\}^n \).

Next, we prove that \( \phi(x) = \min_{a \in D_\varphi}(\phi(a) + \|x - a\|_1) \) for all \( x \in \{0,1\}^n \). Inequality (9) implies \( \phi(x) \leq \phi(w) + \|x - w\|_1 \) for any \( x \) and \( w \) in \( \{0,1\}^n \). So, since we can take \( w \) to be any point in \( D_\varphi \), in this inequality, for all \( x \in \{0,1\}^n \) we have \( \phi(x) \leq \min_{a \in D_\varphi}(\phi(a) + \|x - a\|_1) \).

Say that an \( l_1 \)-path \( \langle q_0, q_1, \ldots, q_m \rangle \) in \( \{0,1\}^n \) is \( \phi \)-decreasing if the sequence \( |\phi(q_i)| \) \( 0 \leq i \leq m \) is decreasing. Let \( \pi = \langle x = p_0, p_1, \ldots, p_k \rangle \) be a \( \phi \)-decreasing \( l_1 \)-path in \( \{0,1\}^n \) starting from \( x \) that is maximal in the sense that it is not an initial segment of any other such \( l_1 \)-path. The length \( k \) of \( \pi \) is at least \( \|x - p_k\|_1 \) and, since \( \pi \) is \( \phi \)-decreasing, \( \phi(p_k) = \phi(x) - k \leq \phi(x) - \|x - p_k\|_1 \). Hence \( \phi(x) \geq \phi(p_k) + \|x - p_k\|_1 \). As \( p_k \in D_\varphi \) (otherwise \( p_k \) would not be an \( l_1 \)-local minimum of \( \phi \) and so \( \pi \) would not be maximal) this inequality implies \( \phi(x) \geq \min_{a \in D_\varphi}(\phi(a) + \|x - a\|_1) = \min_{a \in D_\varphi}(\phi(a) + \|x - a\|_1) \) for all \( x \in \{0,1\}^n \).

Now suppose \( \varphi' \) is an admissible labeling of \( \{0,1\}^n \) such that \( \phi(x) = \min_{a \in D_\varphi}(\varphi'(a) + \|x - a\|_1) \) for all \( x \in \{0,1\}^n \). We need to show that \( \varphi' = \varphi \). We do this by showing

(a) \( \varphi' = \varphi \) on \( D_{\varphi'} \).

(b) Every point in \( D_{\varphi'} \) is an \( l_1 \)-local minimum of \( \varphi \).

(c) No point in \( \{0,1\}^n \setminus D_{\varphi'} \) is an \( l_1 \)-local minimum of \( \varphi \).

Suppose \( x \in D_{\varphi'} \). If \( a = x \), then \( \varphi'(a) + \|x - a\|_1 = \varphi'(x) \). If \( a \in D_{\varphi'} \setminus \{x\} \), then condition 3 in the definition of an admissible labeling implies that

\[
\varphi'(a) + \|x - a\|_1 \geq \varphi'(x) + 2.
\]

Hence \( \phi(x) = \min_{a \in D_{\varphi'}}(\varphi'(a) + \|x - a\|_1) = \varphi'(x) \). This establishes (a).
Let $y$ be any $l_1$-neighbor in $\{0,1\}^n$ of the point $x$ in $D_{\varphi'}$. Then, for all $a \in D_{\varphi'} \setminus \{x\}$, $\varphi'(a) + \|y - a\|_1 \geq \varphi'(a) + \|x - a\|_1 - 1 \geq \varphi'(x) + 1$ by (10). Also, if $a = x$ then $\varphi'(a) + \|y - a\|_1 = \varphi'(x) + 1$. Hence $\phi(y) = \min_{a \in D_{\varphi'}} (\varphi'(a) + \|y - a\|_1) = \varphi'(x) + 1 = \phi(x) + 1$. This establishes (b).

Now let $w$ be any point in $\{0,1\}^n \setminus D_{\varphi'}$. Then there is some point $p \in D_{\varphi'}$ such that

$$\phi(w) = \min_{a \in D_{\varphi'}} (\varphi'(a) + \|w - a\|_1) = \varphi'(p) + \|w - p\|_1.$$ (11)

Let $\pi = (p = p_0, p_1, \ldots, p_k = w)$ be a shortest $l_1$-path in $\{0,1\}^n$ from $p$ to $w$. Then $\|p_k - p\|_1 = \|w - p\|_1 - 1$, so $\phi(p_k - p) = \min_{a \in D_{\varphi'}} (\varphi'(a) + \|p_k - a\|_1) \leq \varphi'(p) + \|p_k - p\|_1 = \phi(w) - 1$ by (11). Hence $w$ is not an $l_1$-local minimum of $\phi$ (as $p_k - p$ is an $l_1$-neighbor of $w$). This establishes (c) and completes the proof of Theorem 7.1.

7.3. Proof of Theorem 7.2

It follows from condition 4 of the definition of an admissible labeling that there is a point $a_0 \in D_{\varphi}$ such that $\varphi(a_0) = 0$. However, $\varphi(a_0) = 0$ implies $\phi(a_0) = 0$ too, so we have shown that $\phi(a) = 0$ for some $a \in \{0,1\}^n$.

Let $x$ and $y$ be arbitrary $l_1$-neighbors in $\{0,1\}^n$. To confirm that $\phi$ is a normalized-admissible function, we must show that $|\phi(x) - \phi(y)| = 1$. For every $a \in D_{\varphi}$, $|\phi(a) + \|x - a\|_1 - (\varphi(a) + \|y - a\|_1)| \leq \|x - y\|_1 = 1$. Hence $|\phi(x) - \phi(y)| = |\min_{a \in D_{\varphi}} (\varphi(a) + \|x - a\|_1) - \min_{a \in D_{\varphi}} (\varphi(a) + \|y - a\|_1)| \leq 1$. We now show that $\phi(x)$ and $\phi(y)$ have opposite parity, so that $|\phi(x) - \phi(y)| = 1$.

For all $a_1, a_2 \in D_{\varphi}$,

$$(\varphi(a_1) + \|x - a_1\|_1) - (\varphi(a_2) + \|x - a_2\|_1)$$

$$\equiv \|a_1 - a_2\|_1 + \|x - a_1\|_1 - \|x - a_2\|_1 \pmod{2} \text{ by condition 2}$$

$$\equiv \|a_1 - a_2\|_1 + \|x - a_1\|_1 + \|x - a_2\|_1 \pmod{2}.$$ We claim that $\|a_1 - a_2\|_1 + \|x - a_1\|_1 + \|x - a_2\|_1 \equiv 0 \pmod{2}$. Indeed, the $l_1$-adjacency graph of $\{0,1\}^n$—i.e., the graph with vertex set $\{0,1\}^n$ in which two vertices are adjacent if and only if they are $l_1$-adjacent—is bipartite. (Its two vertex classes, respectively, consist of the points for which the sum of the coordinates is even, and the points for which the sum of the coordinates is odd.) So every circuit in this graph is of even length. As there is such a circuit of length $\|a_1 - a_2\|_1 + \|x - a_1\|_1 + \|x - a_2\|_1$ that starts at $a_2$, runs through $a_1$ and then $x$, and returns to $a_2$, $\|a_1 - a_2\|_1 + \|x - a_1\|_1 + \|x - a_2\|_1$ is even as we claimed.

Thus, for all $a_1, a_2 \in D_{\varphi}$, $(\varphi(a_1) + \|x - a_1\|_1)$ has the same parity as $(\varphi(a_2) + \|x - a_2\|_1)$; and $\min_{a \in D_{\varphi}} (\varphi(a) + \|x - a\|_1)$ must have this same parity. Similarly, for all $a_1, a_2 \in D_{\varphi}$, $(\varphi(a_1) + \|y - a_1\|_1)$ has the same parity as $(\varphi(a_2) + \|y - a_2\|_1)$; and $\min_{a \in D_{\varphi}} (\varphi(a) + \|y - a\|_1)$ must have this same parity. As $(\varphi(a_1) + \|x - a_1\|_1)$ and $(\varphi(a_1) + \|y - a_1\|_1)$ have opposite parity, it follows that $\phi(x) = \min_{a \in D_{\varphi}} (\varphi(a) + \|x - a\|_1)$
7.4. Isomorphism classes of admissible labelings

The bijection of the admissible labelings \( \varphi \) of \( \{0,1\}^n \) onto the normalized-admissible functions \( \phi \) on \( \{0,1\}^n \) that is given by

\[
\phi(x) = \min_{a \in \mathcal{D}_\varphi} (\varphi(a) + \|x-a\|_1)
\]

induces a bijection of the isomorphism classes of admissible labelings of \( \{0,1\}^n \) onto the isomorphism classes of normalized-admissible functions on \( \{0,1\}^n \). This follows from:

**Theorem 7.3.** Let \( \phi_1 \) and \( \phi_2 \) be normalized-admissible functions on \( \{0,1\}^n \) and let \( \varphi_1 \) and \( \varphi_2 \) be admissible labelings of \( \{0,1\}^n \) such that, for \( i = 1, 2 \),

\[
\phi_i(x) = \min_{a \in \mathcal{D}_{\varphi_i}} (\varphi_i(a) + \|x-a\|_1)
\]

for all \( x \in \{0,1\}^n \). Let \( \psi \) be an isometry of \( \{0,1\}^n \) onto itself. Then \( \psi \) is an isomorphism of \( \varphi_1 \) to \( \varphi_2 \) if and only if \( \psi \) is an isomorphism of \( \varphi_1 \) to \( \varphi_2 \).

**Proof.** \( \psi \) is an isomorphism of \( \varphi_1 \) to \( \varphi_2 \) if and only if \( \varphi_1 = \varphi_2 \circ \psi \), and by (13) the latter is true if and only if

\[
\forall x \in \{0,1\}^n \min_{a \in \mathcal{D}_{\varphi_1}} (\varphi_1(a) + \|x-a\|_1) = \min_{a \in \mathcal{D}_{\varphi_2}} (\varphi_2(a) + \|\psi(x) - a\|_1).
\]

Since \( \psi \) is an isometry of \( \{0,1\}^n \) onto itself, \( \psi \) and \( \psi^{-1} \) map \( l_1 \)-paths to \( l_1 \)-paths of the same length. So the right-hand side of (14) is equal to \( \min_{a \in \mathcal{D}_{\varphi_2}} (\varphi_2(a) + \|x - \psi^{-1}(a)\|_1) \) and hence equal to \( \min_{a \in \mathcal{D}_{\varphi_2}} (\varphi_2(\psi(a)) + \|x-a\|_1) \). Thus \( \psi \) is an isomorphism of \( \varphi_1 \) to \( \varphi_2 \) if and only if

\[
\forall x \in \{0,1\}^n \min_{a \in \mathcal{D}_{\varphi_1}} (\varphi_1(a) + \|x-a\|_1) = \min_{a \in \mathcal{D}_{\varphi_2}} (\varphi_2(\psi(a)) + \|x-a\|_1).
\]

Now (15) is evidently true if \( \psi \) is an isomorphism of \( \varphi_1 \) to \( \varphi_2 \) (by our definition of isomorphism). Conversely, suppose (15) is true. Let \( \varphi' \) be the function such that \( \mathcal{D}_{\varphi'} = \psi^{-1}[\mathcal{D}_{\varphi}] \) and \( \varphi' = \varphi_2 \circ \psi \) on \( \mathcal{D}_{\varphi'} \). Then \( \varphi' \) is an admissible labeling of \( \{0,1\}^n \) and \( \psi \) is an isomorphism of \( \varphi' \) to \( \varphi_2 \). By (13) and (15), \( \phi_1(x) = \min_{a \in \mathcal{D}_{\varphi_1}} (\varphi_1(a) + \|x-a\|_1) = \min_{a \in \mathcal{D}_{\varphi_2}} (\varphi_2(a) + \|x-a\|_1) \) for all \( x \in \{0,1\}^n \). Hence, by Theorem 7.1, \( \varphi_1 = \varphi' \) and so \( \psi \) is an isomorphism of \( \varphi_1 \) to \( \varphi_2 \). \( \square \)

7.5. \( \{0,1\}\)-admissible labelings

An admissible labeling \( \varphi \) will be called a \( \{0,1\}\)-admissible labeling if \( \varphi(x) \in \{0,1\} \) for all \( x \in \mathcal{D}_{\varphi} \); otherwise \( \varphi \) will be called a non-\( \{0,1\}\)-admissible labeling. When \( n \leqslant 3 \)
it follows from conditions 3 and 4 of the definition of an admissible labeling that every admissible labeling of \( \{0, 1\}^n \) is a \( \{0, 1\} \)-admissible labeling. Conditions 3 and 4 similarly imply that every admissible labeling \( \varphi \) of \( \{0, 1\}^4 \) is a \( \{0, 1\} \)-admissible labeling, except in the case where the domain of \( \varphi \) consists of two diametrically opposite points in \( \{0, 1\}^4 \) and one of these points is mapped to 0 by \( \varphi \) while the other is mapped to 2.

For any positive integer \( n \), the \( \{0, 1\} \)-admissible labelings of \( \{0, 1\}^n \) are easily identified. Indeed, let \( M \) be a non-empty subset of \( \{0, 1\}^n \) such that no two points in \( M \) are \( l_1 \)-adjacent. Then it follows easily from conditions 2 and 4 that every \( \{0, 1\} \)-admissible labeling of \( \{0, 1\}^n \) with domain \( M \) is given by the appropriate one of the following two rules:

**Rule A:** If \( \|a - b\|_1 \) is even for all \( a, b \in M \), then the function with domain \( M \) that maps all points of \( M \) to 0 is a \( \{0, 1\} \)-admissible labeling, and is the only \( \{0, 1\} \)-admissible labeling with domain \( M \).

**Rule B:** If \( \|a - b\|_1 \) is odd for some \( a \) and \( b \) in \( M \), and \( M_1 \) and \( M_2 \) are the equivalence classes of the relation \( \{(x, y) \mid \|x - y\|_1 \text{ is even}\} \) on \( M \), then the characteristic functions \( \varphi_1 \) and \( \varphi_2 \) of \( M_1 \) and \( M_2 \) on \( M \) (i.e., the functions \( \varphi_i \) \((i = 1, 2)\) on \( M \) such that \( \varphi_i(x) = 1 \text{ or } 0 \) according to whether \( x \in M_i \) or \( x \not\in M_i \)) are \( \{0, 1\} \)-admissible labelings, and are the only two \( \{0, 1\} \)-admissible labelings with domain \( M \).

7.6. A way to find all admissible \( l_1 \)-connected topological adjacencies on \( \mathbb{Z}^n \)

It is now clear that the problem of identifying all isomorphism classes of admissible \( l_1 \)-connected topological adjacencies on \( \mathbb{Z}^n \) can be solved in the following way:

**Step 1:** Find all symmetry classes of non-empty subsets \( M \) of \( \{0, 1\}^n \) such that no two points in \( M \) are \( l_1 \)-adjacent, and choose one set \( M \) from each symmetry class. (Two subsets of \( \{0, 1\}^n \) are considered to belong to the same symmetry class if there is an isometry of \( \{0, 1\}^n \) onto itself that maps one subset onto the other. Recall that all isometries of \( \{0, 1\}^n \) to itself are given by Fact 5.2.)

**Step 2:** For each of the sets \( M \) chosen in Step 1, find the admissible labeling or labelings with domain \( M \). When there is more than one labeling with the same domain \( M \), some of the labelings may be isomorphic to each other; if so, discard all but one labeling from each isomorphism class.

**Step 3:** For each admissible labeling \( \varphi \) found (but not discarded) in Step 2, let \( \varphi_0 \) be the normalized-admissible function that corresponds (via (12)) to \( \varphi \), and let \( \tilde{\varphi} \) be the admissible labeling that corresponds to \( \varphi \). If \( \tilde{\varphi} \) is not isomorphic to \( \varphi \), then \( \tilde{\varphi} \) is isomorphic to a different labeling \( \varphi' \) found (but not discarded) in Step 2; in such cases discard just one of the labelings \( \varphi \) and \( \varphi' \).

Each of the admissible labelings that remain after Step 3 determines an admissible \( l_1 \)-connected topological adjacency on \( \mathbb{Z}^n \) via (12) and (6) (in Theorem 6.2). Every admissible \( l_1 \)-connected topological adjacency on \( \mathbb{Z}^n \) is isomorphic to exactly one of the adjacencies obtained in this way.
For \( n \leq 4 \) the labelings to be found in Step 2 are all given by rules A and B above, with the single exception noted above in the case \( n = 4 \). Ignoring this exceptional case, when Rule A applies there is just one admissible labeling with domain \( M \). When Rule B applies the two labelings given by the rule may or may not be isomorphic; if they are isomorphic then one of the two will be discarded in Step 2.

### 7.7. Results for \( n = 2, 3 \) and 4

When \( n = 2 \) just two subsets \( M \) of the unit lattice square \( \{0,1\}^2 \) are produced by Step 1: a singleton set, and a set of two 8-adjacent (but not 4-adjacent) points. In both cases Rule A applies, so there is just one admissible labeling \( \varphi \) for each \( M \). In Step 3, we find that each of these two labelings \( \varphi \) is isomorphic to \( \tilde{\varphi} \), so no labeling is discarded. Hence there are exactly two isomorphism classes of admissible \( l_1 \)-connected topological adjacencies on \( \mathbb{Z}^2 \). The corresponding adjacencies (obtained by applying Theorems 7.2 and 6.2 to the two \( \varphi \)'s) are exactly those shown in Fig. 1 above.

When \( n = 3 \), Step 1 produces just five subsets \( M \) of the unit lattice cube \( \{0,1\}^3 \):

1. a singleton set,
2. a set of two 18-adjacent (but not 6-adjacent) points,
3. a set of three pairwise 18-adjacent (but not 6-adjacent) points,
4. a set of four pairwise 18-adjacent (but not 6-adjacent) points,
5. a set of two 26-adjacent (but not 18-adjacent) points.

In the first four cases Rule A applies, so that there is just one admissible labeling \( \varphi \) in each case. In case 5, Rule B applies, so there are two admissible labelings; but they are isomorphic, so one of the two is discarded. Thus Step 2 produces just one admissible labeling for each of the five cases. For \( 1 \leq i \leq 5 \), let \( \varphi_i \) denote the labeling for case \( i \). In Step 3 we find that, for \( i = 1, 2 \) and 4, \( \varphi_i \) is isomorphic to \( \tilde{\varphi}_i \). But \( \tilde{\varphi}_2 \) is isomorphic to \( \varphi_5 \), so one of \( \varphi_3 \) and \( \varphi_5 \) must be discarded, leaving us with just four admissible labelings. Hence, there are exactly four isomorphism classes of admissible \( l_1 \)-connected topological adjacencies on \( \mathbb{Z}^3 \). The corresponding adjacencies (obtained by applying Theorems 7.2 and 6.2 to the four undiscarded \( \varphi \)'s) are those shown in Fig. 2 above.

When \( n = 4 \), Step 1 produces just 20 subsets \( M \) of \( \{0,1\}^4 \). Rule A applies to 15 of the subsets: a singleton set, two 2-point sets, two 3-point sets, four 4-point sets, two 5-point sets, two 6-point sets, a 7-point set, and an 8-point set. Rule B applies to the other five subsets: a 2-point set, a 3-point set, two 4-point sets, and a 5-point set. Rule A gives one labeling for each of the first 15 sets. Rule B gives two labelings for each of the other five sets; but in the cases of the 2-point set and one of the two 4-point sets that Rule B applies to, the two labelings are isomorphic and so in each case one of the labelings must be discarded. So there are just eight Rule B labelings. This gives \( 15 + 8 = 23 \) non-isomorphic \( \{0,1\} \)-admissible labelings. As was mentioned in the first paragraph of Section 7.5, there is also a non-\( \{0,1\} \)-admissible labeling. Thus, we have a total of 24 admissible labelings after Step 2. Let \( \langle \varphi_i \mid 1 \leq i \leq 24 \rangle \) be an enumeration of these labelings.
Fig. 4. The case $n = 4$ of Section 7.7: The 15 $\varphi$'s of Step 3 that are given by Rule A. $\cong$ denotes isomorphism. ($\varphi_{16} - \varphi_{23}$ are shown in Fig. 5, $\varphi_{24}$ in Fig. 6.) Each (•) denotes a point where $\varphi_i$ has value 0. Each (○) denotes a point in $\{0, 1\}^4$ that is not in the domain of $\varphi_i$.

Figs. 4–6 show the 24 $\varphi$'s and also show for each $i$ the value of $j$ such that $\overline{\varphi_j}$ is isomorphic to $\varphi_i$. $\cong$ denotes isomorphism. In each diagram the 16 vertices represent the 16 points of $\{0, 1\}^4$; two vertices are joined by an edge if and only if they represent $l_1$-adjacent points.
In Step 3 we find there are just eight unordered pairs \( \{i, j\}, \ i \neq j, \) such that \( \overline{\varphi_j} \) is isomorphic to \( \varphi_i; \) in each case either \( \varphi_i \) or \( \varphi_j \) must be discarded. (For the other eight \( i \)'s, \( \overline{\varphi_i} \) is isomorphic to \( \varphi_i. \) ) So \( 24 - 8 = 16 \) admissible labelings remain after Step 3. The reader can obtain the corresponding adjacencies by applying Theorems 7.2 and 6.2.
Fig. 6. The case $n = 4$ of Section 7.7 (continued): $\varphi_{24}$ is a non-$\{0,1\}$-admissible labeling of $\{0,1\}^4$, and the only such labeling up to isomorphism. $\cong$ denotes isomorphism. ($\varphi_8$ is shown in Fig. 4.) ($\bullet$) and ($\bigcirc$) denote points where $\varphi_{24}$ has value 0 and 2, respectively. Each ($\circ$) denotes a point in $\{0,1\}^4$ that is not in the domain of $\varphi_{24}$.

to these 16 admissible labelings. We conclude that there are exactly 16 isomorphism classes of admissible $l_1$-connected topological adjacencies on $\mathbb{Z}^4$.

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