Minimal non-simple sets in 4D binary images

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Abstract

One way to verify that a proposed parallel thinning algorithm “preserves topology” is to check that no iteration can ever delete a minimal non-simple ("MNS") set. This is a practical verification method because few types of set can be MNS without being a component. Ronse, Hall, Ma, and the authors have solved the problem of finding all such types of set for 2D and 3D Cartesian grids, 2D hexagonal grids, and 3D face-centered cubic grids. Here we solve this problem for a 4D Cartesian grid, in the case where 80-adjacency is used on 1’s and 8-adjacency on 0’s.

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1. Introduction

1.1. Overview of the paper

This paper deals with \textit{n}-dimensional binary images, especially 4D binary images. Our main purpose is to state and prove a theorem that can be used to establish the “topological soundness” of parallel thinning algorithms for 4D binary images. The concept of topological soundness we apply in this paper to 4D thinning algorithms is consistent with the use of 80-connectedness to define connected subsets of the 4D
binary image that is being thinned and the use of 8-connectedness to define connected subsets of its complement. (Here 80- and 8-connectedness are the 4D analogs of the 3D concepts of 26- and 6-connectedness; for precise definitions see Section 3.2.)

We use the term \( n \)-xel to mean a closed unit \( n \)-cube in \( \mathbb{R}^k \) (where \( k \geq n \)) whose vertices are lattice points; a 2-xel in \( \mathbb{R}^2 \) is called a pixel, and a 3-xel in \( \mathbb{R}^3 \) is called a voxel. We identify each \( n \)-dimensional binary image with an \( n \)-xel in \( \mathbb{R}^n \), and identify the binary image itself with the set of those \( n \)-cubes. So an \( n \)-dimensional binary image is, for us, simply a finite set of \( n \)-xels in \( \mathbb{R}^n \).

In the 1980’s Ronse [12] introduced the concept of a minimal non-simple (MNS) set of a binary image, which we define in Section 2.2 below. One can verify that any proposed parallel thinning algorithm (or, more generally, any given algorithm of the kind defined in Section 2.3) is topologically sound, in the sense that it “preserves topology” for all possible input images, by verifying that no iteration of that algorithm can ever delete an MNS set. A verification method of this kind is described in Section 2.4.

The main goal of this paper is to prove Theorem 8 in Section 4, which is a 4D generalization of analogous results for 2D and 3D binary images that have been established by Ronse [12], Ma [10], and Kong [7]. Theorem 8 tells us that an MNS set of a 4D binary image must either be a subset of a \( 2 \times 2 \times 2 \) block of eight 4-xels, or be a component of the image that is a subset of a \( 2 \times 2 \times 2 \times 2 \) block of 16 4-xels. This shows that (as in the 2D and 3D cases) there are relatively few types of 4D MNS set. Thus when verifying that no iteration of a proposed 4D thinning algorithm can ever delete an MNS set (so that the algorithm preserves topology) there are relative few types of set that must be considered.

While our mathematical arguments are for the most part discrete, there are places—e.g., the statement of the topology preservation condition \( \text{TPres} \) below—where it is necessary to think in terms of the polyhedron that is the union of all the \( n \)-xels of the \( n \)-dimensional binary image.

1.2. 4D thinning algorithms and our definition of topology preservation

Thinning is the process of simplifying binary images by reducing objects to “thin skeletons.” Thinning algorithms are widely used in 2D and 3D image processing. However, thinning can also be applied to 4- and higher-dimensional binary images.

A time-series of \( n \)-dimensional binary images can be represented by an \( (n+1) \)-dimensional binary image. When such \( (n+1) \)-dimensional images are thinned, analysis of the resulting skeletons can provide information on the motion of bodies in the corresponding time-series of \( n \)-dimensional images. Fig. 1 shows a simple example of this, in the case \( n = 1 \). (Note that the thinned image on the right of Fig. 1 cannot be produced by independently thinning each of the 1D time-slices. Independent thinning of the time-slices will produce a disconnected skeleton—with discontinuities between time-slices 4 and 5, and between time-slices 7 and 8, for example—because each of the time-slices 5, 6, 7, 17, 18, and 19 will be thinned to just one element.) In the case \( n = 3 \), this gives one application of 4D thinning.
Jonker [5] describes a different application of 4- and higher-dimensional thinning. The application is to robot path planning (e.g., planning collision-free motion of a pair of robots in a specified planar region) and, more generally, to the problem of finding a way to move a system from one configuration to another. The configuration space of any system with just \( n \) degrees of freedom may be regarded as a subset of Euclidean \( n \)-space \( \mathbb{R}^n \), and can therefore be represented by an \( n \)-dimensional binary image. Any continuous motion of the system from configuration \( a \) to configuration \( b \) corresponds to a path from \( a \) to \( b \) in the configuration space. Thinning the \( n \)-dimensional binary image of the configuration space is a useful first step in finding such a path.

Thinning algorithms are required to “preserve topology.” We interpret this to mean that a thinning algorithm has to satisfy the following condition:

**TPres** The union of the \( n \)-xels of the output image must always be a strong deformation retract of the union of the \( n \)-xels of the input image.

This condition stipulates that it must be possible to continuously deform the union of the \( n \)-xels of the input image over itself onto the union of the \( n \)-xels of the output image, in such a way that all points in the latter polyhedron remain fixed during the deformation process. (For a precise definition of strong deformation retraction, see, e.g., [11, p. 30].) In the 2D and 3D cases, **TPres** is consistent with well established practice: It is satisfied by existing thinning algorithms that preserve 8-/26-connectedness of objects and 4-/6-connectedness of the background.

This paper addresses the problem of how to verify that a proposed thinning algorithm satisfies **TPres**. The problem, and variants of it, have been studied by a number of researchers (including Rosenfeld [14], Ronse [12], Hall [3], Ma [10], Bertrand [1], and the authors [2,6,7]). In the 2D and 3D cases, one fairly general verification...
method is based on the concept of a minimal non-simple set. We describe one version of this method in Section 2.4. Our main theorem, Theorem 8, makes it possible to apply the same method in the 4D case.

2. Simple and minimal non-simple sets

We now give a more detailed account of the concept of an MNS set, and the relevance of MNS sets in verifying the topological soundness of parallel thinning algorithms (and other iterative local deletion algorithms of the kind defined in Section 2.3).

2.1. Simple n-xels and simple sets

Let \( I \) be an \( n \)-dimensional binary image (i.e., a finite set of \( n \)-xels in \( \mathbb{R}^n \)).

We say that an \( n \)-xel \( q \) is simple in \( I \) if \( q \in I \) and \( \bigcup (I \setminus \{q\}) \) is a strong deformation retract of \( \bigcup I \). In other words, the \( n \)-xel \( q \) is simple in \( I \) if and only if \( q \in I \) and the polyhedron given by the union of all the \( n \)-xels of \( I \) can be continuously deformed over itself onto the polyhedron given by the union of all the \( n \)-xels of \( I \) other than \( q \), in such a way that all points in the latter polyhedron remain fixed throughout the deformation process.

In the case \( n = 2 \) this is one of the oldest concepts of digital topology. It was introduced by Rosenfeld and by Hilditch in the late 1960’s. Rosenfeld’s definition of an 8-deletable pixel in [13] was purely discrete, but mathematically equivalent to our definition of a simple pixel. In [4], Hilditch gave the same definition as we have given (see condition 5 on page 411 of her paper), but rather more informally, and only for 2D images.

In this paper we are mainly interested in the case \( n = 4 \). Our work will be based on an (essentially) discrete characterization of simple 4-xels in 4D binary images that was given by the second author in [8]. This characterization will be stated as Theorem 2 below. It is a generalization to 4D images of a similar characterization of simple voxels (of 3D images) that was given in [7].

Whereas other authors (with the exception of Hilditch) have used discrete characterizations of simple pixels and voxels to define the concept of simpleness in 2D and 3D images, the definition given above involves continuous deformation. But our definition of simpleness has the good property of being independent of dimensionality. It is also independent of the shapes of xels. For example, our definition is applicable to 3D binary images on a face-centered cubic grid [2], except that instead of cubical xels we then use rhombic dodecahedra belonging to a tessellation of 3-space by congruent rhombic dodecahedra. (These are the Voronoi neighborhoods of the grid points of a face-centered cubic grid. See Fig. 2.)

A set \( S \) of \( n \)-xels is said to be simple in the image \( I \) if the elements of \( S \) can be arranged in a sequence in which each element is simple after all of its predecessors in the sequence have been removed from \( I \). Thus \( S \) is simple in \( I \) if and only if \( S = \emptyset \), or there is an enumeration \( s_1, s_2, \ldots, s_k \) of the \( n \)-xels of \( S \) such that \( s_1 \) is simple in \( I \) and
each $n$-xel $s_i$ ($1 < i \leq k$) is simple in the binary image $I \setminus \{s_j | j < i\}$. Note that if $S$ is simple in $I$ then $S \subseteq I$. Note also that, for any $n$-xel $p$, $\{p\}$ is simple in $I$ if and only if $p$ is simple in $I$.

### 2.2. Minimal non-simple sets

An MNS set of the $n$-dimensional binary image $I$ is a set $N \subseteq I$ such that $N$ is not simple in $I$, but all proper subsets of $N$ are simple in $I$. This concept was introduced by Ronse in [12], though Ronse called such sets “minimal non-deletable.”

A trivial example of an MNS set of $I$ is a singleton set consisting of a non-simple $n$-xel of $I$. For a less trivial 2D example of an MNS set, consider a 2D binary image that consists just of a $2 \times 3$ block of 6 pixels. The two central pixels constitute an MNS set of this image: Each of the pixels is simple in the image, but becomes non-simple if the other pixel is removed from the image.

It is also not hard to give a non-trivial example of an MNS set in a 4D binary image. First note that an $n$-xel is the trajectory of an $(n-1)$-xel as it moves one unit in a direction that is perpendicular to the $(n-1)$-xel. (Indeed, the reader may sometimes find it helpful to think of 4-xels as trajectories of 3-xels that move through a unit of time.) Thus we can draw pictures of 4-xels using the representation shown in Fig. 3. Using this representation of 4-xels, we give an example of a 4D MNS set in Fig. 4.

We say that a set $D$ of $n$-xels in $\mathbb{R}^n$ can be MNS if there is some $n$-dimensional binary image $J$ such that $D$ is an MNS set of $J$. Similarly, we say that a set $D$ of $n$-xels in $\mathbb{R}^n$ can be MNS without being a component if there is some $n$-dimensional binary image $J$ such that $D$ is an MNS set of $J$ but $\bigcup D$ is not a connected component of $\bigcup J$. Thus $D$ can be MNS without being a component if and only if there is some

![Fig. 3. An $n$-xel can be viewed as the trajectory of a moving $(n-1)$-xel.](image)
2.3. Iterative local deletion algorithms and topology preserving local deletion conditions

Most thinning algorithms in the literature are iterative local deletion algorithms. These are the only kinds of thinning algorithms we shall consider in this paper.

By an iterative local deletion algorithm (for \( n \)-dimensional binary images) we mean an algorithm in which each iteration removes from the image, in parallel, all the \( n \)-xels that satisfy a specified local deletion condition. Here the term local deletion condition (for \( n \)-dimensional binary images) means a predicate \( \mathbb{P}(p, I) \), defined on \( n \)-xels \( p \) in \( \mathbb{R}^n \) and \( n \)-dimensional binary images \( I \), with the following three properties:

(A) \( \mathbb{P}(p, I) \Rightarrow p \in I \).

(B) \( \mathbb{P} \) is local in the following sense: there exists a positive integer \( \rho \) such that, for all \( p \) and \( I \), \( \mathbb{P}(p, I) = \mathbb{P}(p, \{ q \in I \mid \|q - p\|_\infty \leq \rho \}) \). (For local deletion conditions used by thinning algorithms, this typically holds for \( \rho = 2 \), and quite commonly holds for \( \rho = 1 \). Here \( \|q - p\|_\infty \) denotes the maximum of the absolute values of the components of the vector from the centroid of \( p \) to the centroid of \( q \).)

(C) \( \mathbb{P} \) is translation-invariant in the following sense: For all \( p \) and \( I \), and all \( n \)-dimensional integer vectors \( v \), \( \mathbb{P}(p + v, I + v) = \mathbb{P}(p, I) \).

We shall say that a local deletion condition \( \mathbb{P} \) for \( n \)-dimensional binary images is topology preserving if for every \( n \)-dimensional binary image \( I \) the set \( \{ p \in I \mid \mathbb{P}(p, I) \} \) is simple in \( I \).

If an iterative local deletion algorithm runs for at least \( k \) iterations, and \( I_k \) and \( I_{k+1} \), respectively, denote the image at the start and the end of the \( k \)th iteration, then

\[
I_{k+1} = I_k \setminus \{ p \in I_k \mid \mathbb{P}_k(p, I_k) \} = \{ p \in I_k \mid \text{not } \mathbb{P}_k(p, I_k) \},
\]

where \( \mathbb{P}_k \) denotes the local deletion condition applied at the \( k \)th iteration.

Now a straightforward consequence of our definition of a simple set is that, if a set \( S \) is simple in \( I \), then \( \bigcup (I \setminus S) \) is a strong deformation retract of \( \bigcup I \). So to show that a given iterative local deletion algorithm (such as a proposed thinning algo-
rithm) satisfies our topology preservation condition \( \text{TPres} \), it suffices to establish that 
(with the above definitions of \( P_k \) and \( I_k \)) for each iteration number \( k \) the set 
\( \{ p \in I_k \mid P_k(p, I_k) \} \) that is deleted at the \( k \)th iteration of the algorithm must always 
be simple in the image \( I_k \). Equivalently, it suffices to establish the following:

**TP** For each iteration number \( k \), the local deletion condition \( P_k \) is topology pre-
serving.

In the next section we describe a method of establishing this.

2.4. A method of verifying that a proposed thinning algorithm satisfies \( \text{TPres} \)

Let \( T \) be any iterative local deletion algorithm and, for each iteration number \( k \), 
let \( P^T_k \) be the local deletion condition applied by \( T \) at the \( k \)th iteration. Since any non-
simple set of an \( n \)-dimensional binary image \( J \) must evidently contain an MNS set of 
\( J \), we can show that the local deletion condition \( P^T_k \) is topology preserving by show-
ing that “no MNS set can be completely deleted by \( P^T_k \),” a condition that can be 
more precisely stated as follows: for all sets of \( n \)-xels \( R \) and all \( n \)-dimensional binary 
images \( J \), \( \{ p \in R \mid P^T_k(p, J) \} \) holds for all \( p \in R \), then \( R \) is not MNS in \( J \). So to prove that \( T \) 
satisfies condition TP and hence satisfies our topology preservation condition \( \text{TPres} \), 
it is enough to verify the following condition:

**MNSPres** For all iteration numbers \( k \), all sets \( R \) of \( n \)-xels in \( R^n \), and all \( n \)-dimen-
sional binary images \( J \),

\[
\bigwedge_{p \in R} P^T_k(p, J) \implies R \text{ is not MNS in } J.
\]

This is a fairly general method of verifying that \( T \) satisfies \( \text{TPres} \): while it is possible 
for an iterative local deletion algorithm \( T \) to satisfy \( \text{TPres} \) but not \( \text{MNSPres} \),\(^1\) it is 
safe to say that the vast majority of thinning algorithms in the literature that cor-
rectly satisfy \( \text{TPres} \) also satisfy \( \text{MNSPres} \). The practicality of this verification 
method depends on the fact that not all sets of \( n \)-xels \( R \) have to be considered in 
verifying \( \text{MNSPres} \). Indeed, relatively few sets of \( n \)-xels can be MNS, and even fewer 
can be MNS without being a component.

Say that two sets of \( n \)-xels are of the same type if one set is a translate of the other. 
Suppose we can identify:

\(^1\) Here is a 2D example of an iterative local deletion algorithm \( T \) that satisfies \( \text{TPres} \) but not \( \text{MNSPres} \). 
Let \( P \) be the 2D local deletion condition which deletes all simple north border pixels that have two or more 
8-neighbors in the image. Let \( P^2 \) be the local deletion condition that deletes all the pixels that are deleted 
when \( P \) is applied twice in succession. More precisely, let \( P^2 \) be defined by \( P^2(p, I) \iff P(p, I) \lor P \{ p, I \setminus \{ q \in I \mid P(q, I) \} \} \). Now let \( T \) be the iterative local deletion algorithm that applies the deletion 
condition \( P^2 \) at each iteration. It is well known [14] that \( P \) is topology preserving, so that \( P^2 \) is also 
topology preserving. Hence \( T \) satisfies TP and therefore satisfies \( \text{TPres} \). But if \( J \) is a \( 3 \times 3 \) square of 9 pixels 
and \( p \) is the central pixel of \( J \), then \( P^2(p, J) \) holds and yet \( \{ p \} \) is MNS in \( J \) (since \( p \) is non-simple in \( J \)). So \( T \) 
violates \( \text{MNSPres} \) for this image \( J \) when \( R = \{ p \} \).
(A) All types of sets that can be MNS. (Call these Category A sets.)
(B) All types of sets that can be MNS without being a component. (Call these Category B sets. Note that every Category B set is also a Category A set.)

Then we claim that MNSPres holds if and only if the following two conditions both hold:

1. For all iteration numbers \( k \) and all \( n \)-dimensional binary images \( J \),

\[
\text{not } \bigwedge_{p \in R} \mathcal{P}_k^T(p, J),
\]

whenever \( R \) is a Category A set that is not a Category B set and \( \bigcup R \) is a connected component of \( \bigcup J \).

2. For all iteration numbers \( k \) and all \( n \)-dimensional binary images \( J \),

\[
\bigwedge_{p \in R} \mathcal{P}_k^T(p, J)
\]

\( \Rightarrow R \) is not MNS in \( J \),

whenever \( R \) is a Category B set.

Evidently, if MNSPres holds then its special case (2) holds. Moreover, if MNSPres holds then (1) holds too: for we already know that if MNSPres holds then \( T \) satisfies condition TP, which implies that \( \mathcal{P}_k^T \) cannot delete the whole of \( R \) from \( J \) if \( \bigcup R \) is a connected component of \( \bigcup J \).

To see that MNSPres holds if (1) and (2) hold, first note that if \( R \) is a set of \( n \)-xels and \( J \) is an \( n \)-dimensional binary image then exactly one of the following must be true:
(a) \( R \) is a Category B set.
(b) \( R \) is a Category A set but not a Category B set, and \( \bigcup R \) is a connected component of \( \bigcup J \).
(c) \( R \) is a Category A set but not a Category B set, and \( \bigcup R \) is not a connected component of \( \bigcup J \).
(d) \( R \) is not a Category A set.

Now suppose conditions (1) and (2) hold. Then MNSPres holds in case (a), by (2). Also, MNSPres holds (vacuously) in case (b), by (1). Finally, MNSPres is trivially valid in cases (c) and (d), for in these cases \( R \) cannot be an MNS set of \( J \) (by the definitions of Category B and Category A).

In fact it can be shown that, if last(\( R \)) is any choice function on Category B sets (i.e., any function such that last(\( R \)) \( \in R \) for each Category B set \( R \)) and Rest(\( R \)) = \( R \setminus \{ \text{last}(R) \} \), then condition 2 is equivalent to the following simpler condition:

(2') For all iteration numbers \( k \) and all \( n \)-dimensional binary images \( J \),

\[
\bigwedge_{p \in R} \mathcal{P}_k^T(p, J)
\]

\( \Rightarrow \) last(\( R \)) is simple in \( J \setminus \text{Rest}(R) \),

whenever \( R \) is a Category B set.
That (2') implies (2) is a straightforward consequence of the definition of an MNS set. One can deduce that (2) implies (2') from Lemma 4 and Theorem 7 below. Thus we can conclude that an iterative local deletion algorithm $T$ satisfies our topology preservation condition TPres if $T$ satisfies (1) and (2').

Note that, in verifying (1) and (2') for a particular iteration number $k$ and a particular set $R$, we need only consider a finite number of different $n$-dimensional binary images $J$. This follows from the localness condition (B) in our definition of a local deletion condition. Indeed, let $q_{P_k}$ be the least positive integer $q$ such that, for all $p$ and $I$, $P_k(p, I) = P_k(p, \{q \in I \ | \ ||q - p||_\infty \leq \rho\})$. Then, for any given $k$ and $R$, the conditions (1) and (2') must hold for all $n$-dimensional binary images $J$ if they hold for every $J$ that is contained in the finite set of $n$-xels $\bigcup_{p \in R} \{q \ | \ ||q - p||_\infty \leq q_{P_k}\}$.

To be able to verify that $T$ satisfies (1) and (2') (and hence satisfies TPres), we want to know which types of sets are in Category A and which types are in Category B. These questions were answered by Ronse [12] in the 2D case, and by Ma [10] in the 3D case. A different derivation of their results was given by Kong [7]. For both $n = 2$ and $n = 3$, it turns out that a set of $n$-xels in $\mathbb{R}^n$ belongs to Category A (can be MNS) if and only if it is a nonempty subset of some $2^n$-block of $n$-xels in $\mathbb{R}^n$, and belongs to Category B (can be MNS without being a component) if and only if it is a nonempty subset of some $2^{n-1}$-block of $n$-xels in $\mathbb{R}^n$. Here a "2"-block means a $2 \times 2$ block of pixels when $n = 2$, and a $2 \times 2 \times 2$ block of voxels when $n = 3$. Similarly, a "2"-block means a $2 \times 1$ or $1 \times 2$ block of pixels when $n = 2$, and a $1 \times 2 \times 2$, $2 \times 1 \times 2$, or $2 \times 2 \times 1$ block of voxels when $n = 3$.

The main result of this paper is that the same is true in the 4D case: with analogous definitions of "2"-block and "2"-block" when $n = 4$, a set of 4-xels in $\mathbb{R}^4$ is in Category A (can be MNS) if and only if it is a nonempty subset of some $2^4$-block of 4-xels in $\mathbb{R}^4$, and is in Category B (can be MNS without being a component) if and only if it is a nonempty subset of some $2^{3}$-block of 4-xels in $\mathbb{R}^4$.

3. Other basic concepts and preliminary results

In the rest of this paper the term 4-xel will mean a 4-xel in $\mathbb{R}^4$ unless otherwise stated.

3.1. Xels, their faces, and xel-complexes

An elementary 0-cell is a singleton set $\{i\}$, where $i$ is an integer. An elementary 1-cell is a closed unit interval $[i, i+1]$ of the real line, where $i$ is an integer. So for $0 \leq n \leq 4$ an $n$-xel in $\mathbb{R}^4$ is just a Cartesian product of $n$ elementary 1-cells and $4 - n$ elementary 0-cells, in some order. If $q$ is an $n$-xel for some $n$, then we say $q$ is a xel. A 0-xel will also be called a vertex and a 1-xel will also be called an edge.

If a xel $y$ is a proper subset of a xel $x$, then we say $y$ is a proper face of $x$, and write $y < x$. A xel $x$ is a Cartesian product of elementary 1-cells and elementary 0-cells, and
each proper face of \( x \) can be obtained from the product by replacing one or more of its elementary 1-cells \([i, i + 1]\) with \( \{i\} \) or \( \{i + 1\} \). A set \( K \) of xels is called a xel-complex if \( K \) is finite and, for every xel \( x \in K \), every proper face of \( x \) is also in \( K \). If \( x \) is a xel and \( y < x \) is a \( k \)-xel then we say \( y \) is a \( k \)-face of \( x \).

3.2. Adjacency and connectedness relations on sets of 4-xels

It is easy to show that, for \( 0 \leq i \leq 3 \), a 4-xel has exactly \( 2^{4-i}C(4, i) \) \( i \)-faces, and that if \( f \) is any \( i \)-face of a 4-xel \( q \) then there is exactly one 4-xel \( q' \) for which \( q \cap q' = f \). So for every 4-xel \( q \) there are exactly \( \sum_{j=i}^{3} 2^{4-j}C(4, j) \) other 4-xels \( q' \) such that \( q \) and \( q' \) share a \( j \)-face for some \( j \geq i \). Accordingly, two 4-xels \( q_1 \) and \( q_2 \) are said to be \( \kappa_i \)-adjacent, where \( \kappa_i = \sum_{j=i}^{3} 2^{4-j}C(4, j) \), if \( q_1 \) and \( q_2 \) share a \( j \)-face for some \( j \geq i \). (Note that \( \kappa_0 = 8 \), \( \kappa_1 = 64 \), \( \kappa_2 = 32 \), and \( \kappa_3 = 8 \).) In particular, two 4-xels are 80-adjacent if and only if they are distinct but intersect, and are 8-adjacent if and only if their intersection is a 3-xel.

For any set \( S \) of 4-xels, two 4-xels \( q_1, q_2 \in S \) are said to be \( \kappa_i \)-connected in \( S \) if they are related by the reflexive transitive closure of the \( \kappa_i \)-adjacency relation on \( S \). This is an equivalence relation on \( S \), and if \( S \) is nonempty then each of its equivalence classes is called a \( \kappa_i \)-component of \( S \).

Note that two 4-xels belong to the same 80-component of a 4D binary image \( I \) if and only if they lie in the same connected component of \( \bigcup I \), and they belong to the same 8-component of the 4-xels in the complement of \( I \) if and only if their interiors lie in the same connected component of \( \mathbb{R}^4 \setminus \bigcup I \). In this sense our definition of a simple 4-xel \( q \) in \( I \) in terms of the polyhedra \( \bigcup I \) and \( \bigcup (I \setminus \{q\}) \), and our interpretation TPres of the topology preservation requirement for 4D thinning algorithms, are consistent with the use of 80-connectedness as the definition of connectedness on the 4-xels in the image and 8-connectedness as the definition of connectedness on the 4-xels in its complement.

If \( q_1 \) and \( q_2 \) are \( \kappa_i \)-adjacent, then each of them is said to be a \( \kappa_i \)-neighbor of the other. For any 4-xel \( q \), we write \( N(q) \) to denote the \( 3 \times 3 \times 3 \times 3 \)-neighborhood of \( q \), by which we mean the set consisting of \( q \) itself and all the 80-neighbors of \( q \).

3.3. Attachment sets and an essentially discrete characterization of simple 4-xels

Theorems 1 and 2 below give essentially discrete sets of necessary and sufficient conditions for a 4-xel \( q \) to be simple in a 4D binary image \( I \). These two theorems depend on the concept of the attachment complex in \( I \) of an \( n \)-xel \( q \) in an \( n \)-dimensional binary image \( I \), which is denoted by \( \text{Attach}(q, I) \) and is defined by

\[
\text{Attach}(q, I) = \{ f \mid f < q \text{ and } f < y \text{ for some } y \in I \setminus \{q\} \}.
\]

The closed polyhedral set \( \bigcup \text{Attach}(q, I) \) is called the attachment set of \( q \) in \( I \). This set, which has also been called the shared boundary (see, e.g., [15]), is the subset of the boundary of the \( n \)-xel \( q \) that is shared with other \( n \)-xels in the image \( I \). It is also given by \( \bigcup \text{Attach}(q, I) = q \cap \bigcup (I \setminus \{q\}) \).
The boundary complex of a \( n \)-xel \( q \), denoted by \( \text{Boundary}(q) \), is the set of all the proper faces of \( q \). Evidently, \( \text{Attach}(q, I) \subseteq \text{Boundary}(q) \).

**Theorem 1.** Let \( q \) be a 4-xel in a 4D binary image \( I \). Then \( q \) is simple in \( I \) if and only if the following all hold:

1. \( \bigcup \text{Attach}(q, I) \) is connected and nonempty.
2. \( \bigcup \text{Boundary}(q) \setminus \bigcup \text{Attach}(q, I) \) is connected and nonempty.
3. \( \bigcup \text{Attach}(q, I) \) is simply connected.

For any xel-complex \( K \) in \( \mathbb{R}^4 \), the Euler characteristic of \( K \) is the integer \( \chi(K) \) defined by

\[
\chi(K) = c_0(K) - c_1(K) + c_2(K) - c_3(K) + c_4(K),
\]

where \( c_n(K) \) is the number of \( n \)-xels in \( K \). If \( P \) is the union of the xels of a xel-complex \( K \), then we define \( \chi(P) \) to be \( \chi(K) \). It can be shown that if \( x \) is any xel then \( \chi(x) = 1 \).

**Theorem 2.** Let \( q \) be a 4-xel in a 4D binary image \( I \). Then \( q \) is simple in \( I \) if and only if the following all hold:

1. \( \bigcup \text{Attach}(q, I) \) is connected.
2. \( \bigcup \text{Boundary}(q) \setminus \bigcup \text{Attach}(q, I) \) is connected.
3. \( \chi(\text{Attach}(q, I)) = 1 \).

Theorems 1 and 2 are Theorems 7 and 9 in [8], except that the definition of a simple 4-xel used in that paper might seem to be more stringent than the definition given above: in [8], a 4-xel \( q \) is said to be simple in a 4D binary image \( I \) if \( \bigcup \text{Attach}(q, I) \) is a strong deformation retract of \( q \).

As is explained in [8], it follows from the main result of [9] that the three conditions of Theorem 1 are equivalent to the three conditions of Theorem 2. So the two theorems are equivalent.

An elementary proof of the “if” parts of these theorems is given in [8]. A shorter proof can be given using results of algebraic topology.\(^2\)

Although the “only if” parts of Theorems 1 and 2 are easy to prove if simple 4-xels are defined as in [8], we have to work a little harder to give a proof for the definition of simpleness used in this paper. However, standard techniques of algebraic topology suffice.\(^3\)

The definition of simpleness used in [8] is in fact equivalent to the definition used in this paper, because both definitions are equivalent to the three conditions of Theorem 1 or 2. An advantage of the definition we are now using is that it involves only 4-xels (and does not involve their attachment sets).

---

\(^2\) The “if” part of Theorem 1 can be deduced from the following two facts. Firstly, a simply connected polyhedron is contractible if its reduced homology groups are trivial [11, Cor. 8.3.11]. Secondly, when one contractible polyhedron contains another contractible polyhedron, the latter is a strong deformation retract of the former [16, Cor. 3.2.5, 1.3.11, 1.4.10, and Thm. 1.4.11]. We apply the first fact to \( \bigcup \text{Attach}(q, I) \), and apply the second to the pair \( (q, \bigcup \text{Attach}(q, I)) \).

\(^3\) Assuming \( q \) is simple in \( I \) one can use the exact homology sequences of the pairs \( (q, \bigcup \text{Attach}(q, I)) \) and \( (\bigcup I, \bigcup \{q\}) \) together with the excision theorem to deduce that the reduced homology groups of \( \bigcup \text{Attach}(q, I) \) are all trivial. The three conditions of Theorem 2 follow from this and the Alexander duality theorem.
It may be worth mentioning here that, while Theorem 2 provides an easy way of determining whether or not a 4-ixel in a 4D binary image is simple, there seems to be no analog of Theorem 2 for 5- and higher-dimensional images. Indeed, for $n \geq 5$ the authors do not currently have a good way to determine if the attachment set of an $n$-ixel in an $n$-dimensional binary image is simply connected.

An important benefit of using characterizations of simple 4-xels in terms of their attachment sets is that while it may be hard to visualize certain sets of 4-xels, there is an easy way to visualize the attachment set of a 4-ixel in an image: the attachment set is a subset of the boundary of the 4-ixel and can be represented in a 3D Schlegel diagram of the boundary. This representation was much used in [8], and it was also most helpful to us in convincing ourselves of the truth of our main theorem before we proved it.

3.4. Properties of MNS sets

We will need the following fundamental characterization of MNS sets:

**Theorem 3.** Let $D$ be a nonempty set of 4-xels in a 4D binary image $I$. Then $D$ is MNS in $I$ if and only if the following conditions both hold:

1. Each element $q \in D$ is non-simple in $I \setminus (D \setminus \{q\})$.
2. Each element $q \in D$ is simple in $I \setminus D'$ whenever $D' \subseteq D \setminus \{q\}$.

This is in fact a case of Prop. 6 in [6]. The analogous theorem for 3D binary images was given by Ma [10], and was stated and proved as Prop. 4.3 in [7].

Theorem 3 can be proved in the same way. It is actually a straightforward matter to show that if $D$ satisfies conditions 1 and 2 then $D$ is MNS in $I$, and that if $D$ is MNS in $I$ then $D$ satisfies condition 1. To show that if $D$ is MNS in $I$ then $D$ satisfies condition 2 is less easy. This implication is a consequence of the following Lemma:

**Lemma 4.** Let $I$ be any $n$-dimensional binary image, and let $D'$ and $D' \cup \{q\}$ be simple subsets of $I$ (where $q \not\in D'$). Then $q$ is simple in $I \setminus D'$.

The authors do not know of any purely discrete proof of this lemma, but it follows from well known properties of strong deformation retraction.\footnote{By [16, Cor. 1.4.10, Thm. 1.4.11, Cor. 3.2.5], $p$ is simple in $I$ if and only if the inclusion map of $\bigcup (I \setminus \{p\})$ in $\bigcup I$ is a homotopy equivalence. So, under the hypotheses of Lemma 4, the inclusion maps $i_1 : \bigcup (I \setminus D') \setminus \{q\} \to \bigcup (I \setminus D')$ and $i_2 : \bigcup (I \setminus D') \to \bigcup I$ are such that $i_2$ and $i_2 \circ i_1$ are homotopy equivalences, which implies that $i_1$ is a homotopy equivalence (since $i_1$ is homotopic to the composition of the homotopy inverse of $i_2$ with $i_2 \circ i_1$) and $q$ is simple in $I \setminus D'$.}

The next three theorems state important properties of MNS sets that are fairly straightforward consequences of Theorem 3. These are 4D analogs of 3D results given in [7,10].
Theorem 5. A set $D$ of 4-xels cannot be MNS unless $D$ is a nonempty subset of some $2 \times 2 \times 2 \times 2$ block of 16 4-xels.

Proof. An empty set is simple in any 4D binary image, and so cannot be MNS. Suppose $D$ is not contained in any $2 \times 2 \times 2 \times 2$ block of 4-xels. Then there exist $p_1, p_2 \in D$ such that $p_2 \notin N(p_1)$. Let $D = \{p_1, p_2, p_3, \ldots, p_k\}$ (so $k \geq 2$), and let $I$ be an arbitrary 4D binary image. Whether or not a 4-xel $p$ is simple in a 4D binary image depends only on the part of that image that lies in $N(p)$; this is an immediate consequence of Theorem 2. So, since $p_2 \notin N(p_1)$, if $p_1$ is non-simple in $I$, then if $p_2 \notin N(p_1)$, then $p_1$ is also non-simple in $I \setminus \{p_1, \ldots, p_k\}$. Thus $D$ and $I$ cannot satisfy both of conditions 1 and 2 of Theorem 3 when $q = p_1$, and so $D$ cannot be MNS in $I$.

Theorem 6. Let $D$ be an MNS set of a 4D binary image $I$, and suppose $D$ is not an 80-component of $I$. Then every element of $D$ has an 80-neighbor in $I$.

Proof. If $S$ is an 80-component of a 4D binary image $I$, and $R$ is any proper subset of $I$, then we say that removal of $R$ from $I$ splits $S$ if $S$ contains two or more 80-components of $I \setminus R$. Now it follows from Theorem 2 that this can never happen if $R$ consists of a single simple 4-xel of $I$, and so it also can never happen if the set $R$ is simple in $I$.

By Theorem 5, the set $D$ lies in a $2 \times 2 \times 2 \times 2$ block of 4-xels and is therefore 80-connected. Since $D$ is 80-connected and is not an 80-component of $I$, $D$ is a proper subset of some 80-component $S$ of $I$. If some $p \in D$ had no 80-neighbor in $S \setminus D$, then removal of $D \setminus \{p\}$ from $I$ would split $S$, which is impossible as $D \setminus \{p\}$ is simple in $I$ (since $D$ is MNS in $I$). This proves the theorem.

Theorem 7. If a set $D$ of 4-xels can be MNS without being a component, then every nonempty subset $D'$ of $D$ can be MNS without being a component.

Proof. If $D$ is MNS in a 4D binary image $I$ and $\emptyset \subset D' \subset D$, then (by Theorem 3) $D'$ is MNS in $I \setminus (D \setminus D')$, and if $D$ is also not an 80-component of $I$, then (by Theorem 6) $D'$ is not an 80-component of $I \setminus (D \setminus D')$.

4. The main theorem

We now state our main theorem, which identifies all sets of 4-xels that can be MNS, and all such sets that can be MNS without being a component:

Theorem 8 (Main Theorem). Let $D$ be a set of 4-xels. Then:
(1) $D$ can be MNS if and only if $D$ is a nonempty subset of some $2 \times 2 \times 2 \times 2$ block of 16 4-xels.
(2) $D$ can be MNS without being a component if and only if $D$ is a nonempty subset of some $2 \times 2 \times 2$ block of 8 4-xels.
Note that there are four types of $2 \times 2 \times 2$ block in $\mathbb{R}^4$: such a block could be a $1 \times 2 \times 2 \times 2$, a $2 \times 1 \times 2 \times 2$, a $2 \times 2 \times 1 \times 2$, or a $2 \times 2 \times 2 \times 1$ block of 4-xels. In the rest of this section we give a proof of Theorem 8.

4.1. Useful results

The purpose of this subsection is to present three results that will be used in our proof of the main theorem.

The first result is the Inclusion–Exclusion Principle for Euler characteristics, which is the following identity:

$$\chi\left(\bigcup_{i=1}^{n} K_i\right) = \sum_{T \subseteq \{1,2,\ldots,n\}, T \neq \emptyset} (-1)^{|T|-1} \chi\left(\bigcap_{i \in T} K_i\right).$$

This holds for arbitrary xel-complexes $K_1, K_2, \ldots, K_n$; it follows from the Inclusion–Exclusion Principle for finite sets and the definition of $\chi(K)$.

The second result is the next proposition, which is related to the following lemma:

**Lemma 9.** Let $P$ be a union of xels and let $x$ be an edge or a 2-xel such that $x \not\subseteq P$ and $\chi(x \cap P) = 1$. Then one of the following is true:

1. $x \cap P$ consists of a single vertex of $x$.
2. $x$ is a 2-xel and $x \cap P$ is one of the four edges of $x$.
3. $x$ is a 2-xel and $x \cap P$ is a union of two edges of $x$ that share a vertex.
4. $x$ is a 2-xel and $x \cap P$ is a union of three of the four edges of $x$.

This lemma is easily verified by considering all possible forms of $x \cap P$. From the lemma it is not hard to deduce Proposition 10 below, which will be used in Section 4.3. We omit the proof of the proposition, but expect most readers will find it intuitively clear that all parts of the proposition are valid in each of the four cases of the lemma.

**Proposition 10.** Let $q$ be a 4-xel. Let $P$ be a union of xels in $\text{Boundary}(q)$ and let $x$ be an edge or a 2-xel in $\text{Boundary}(q)$ such that $\chi(x \cap P) = 1$. Then:

1. $P$ is connected if and only if $P \cup x$ is connected.
2. $\bigcup \text{Boundary}(q) \setminus P$ is connected if and only if $\bigcup \text{Boundary}(q) \setminus (P \cup x)$ is connected.
3. $\chi(P) = \chi(P \cup x)$.

The following proposition is the third result. This will save us a lot of case-checking in Section 4.2:

**Proposition 11.** Let $q$ be a 4-xel, and let $X$ be any nonempty set of xels in $\text{Boundary}(q)$ that satisfies one of the following two conditions:
(A) There is some vertex that belongs to all of the xels in $X$.
(B) $X = Y \cup Z$, where $Y \cap Z \neq \emptyset$, there is some vertex that belongs to all of the xels in $Y$, there is some vertex that belongs to all of the xels in $Z$, and no xel in $Y \setminus Z$ intersects a xel in $Z \setminus Y$.

Then $X$ satisfies the following conditions:

1. $\bigcup X$ is connected.
2. $\text{Boundary}(q) \setminus \bigcup X$ is connected.
3. $\chi(\bigcup X) = 1$.

In fact condition A in this proposition is a special case of B. (Take $Y = Z = X$ in B.)

The proposition can be deduced from Theorem 4.1 in [15].

4.2. Proof of the “if” parts of the main theorem

For any 4-xel $q$ and 4D binary image $I$, let $A(q, I) = \{q \cap x \mid x \in I \setminus \{q\}\} \setminus \{\emptyset\}$. Note that, for any 4D binary images $I_1$ and $I_2$, $A(q, I_1 \cup I_2) = A(q, I_1) \cup A(q, I_2)$, $A(q, I_1 \cap I_2) = A(q, I_1) \cap A(q, I_2)$, and $A(q, I_1 \setminus I_2) = A(q, I_1) \setminus A(q, I_2)$. Note also that if $q \in I$ then $\bigcup A(q, I) = \bigcup \text{Attach}(q, I)$.

To show that the “if” part of assertion 1 of Theorem 8 is valid, let $D$ be a non-empty subset of a $2 \times 2 \times 2 \times 2$ block of 4-xels and let $I$ be a 4D binary image such that $D$ is an 80-component of $I$. We claim $D$ is MNS in $I$. Evidently, $D$ satisfies condition 1 of Theorem 3. It remains to show that $D$ also satisfies condition 2 of Theorem 3. Let $q \in D$ and let $I'$ be obtained from $I$ by deleting any proper subset of the other elements of $D$. We need to show that $q$ is simple in $I'$. Let $X = A(q, I')$. As $D$ is contained in a $2 \times 2 \times 2 \times 2$ block of 4-xels, the central vertex of that block belongs to all the 4-xels in $D$ and hence to all the xels in $X$. So, since $\bigcup X = \bigcup \text{Attach}(q, I')$, it follows from Proposition 11 that the three conditions of Theorem 2 hold with $I'$ in place of $I$. Thus $q$ is simple in $I'$, as required.

To show that the “if” part of assertion 2 is also valid, let $I$ be a $2 \times 2 \times 2 \times 3$ block of 4-xels and let $D$ be its central $2 \times 2 \times 2 \times 1$ block (which is clearly not an 80-component of $I$). We claim $D$ is MNS in $I$. If we can prove this then, by Theorem 7, the “if” part of assertion 2 is valid. By symmetry we may assume that $I = \{i_1 \times i_2 \times i_3 \times i_4 \mid i_1, i_2, i_3 \in \{0, 1\}, i_4 \in \{0, 1, 2\}\}$, so that $D = \{i_1 \times i_2 \times i_3 \times [1, 2] \mid i_1, i_2, i_3 \in \{0, 1\}, [1, 2]\}$. $D$ evidently satisfies condition 1 of Theorem 3. To show that $D$ also satisfies condition 2 of Theorem 3, let $q \in D$ and let $I'$ be obtained from $I$ by deleting any proper subset of the other seven 4-xels in $D$. We need to show that $q$ is simple in $I'$.

Let $D^- = \{i_1 \times i_2 \times i_3 \times [0, 1] \mid i_1, i_2, i_3 \in \{0, 1\}\}$ and let $D^+ = \{i_1 \times i_2 \times i_3 \times [2, 3] \mid i_1, i_2, i_3 \in \{0, 1\}\}$. Let $X = A(q, I')$. Since $I' \subseteq I = D^- \cup D \cup D^+$, we have $X = Y \cup Z$, where $Y = A(q, I' \cap (D \cup D^-))$ and $Z = A(q, I' \cap (D \cup D^+))$. As $D \cup D^- = \{i_1 \times i_2 \times i_3 \times i_4 \mid i_1, i_2, i_3, i_4 \in \{0, 1\}\}$, the vertex $(1, 1, 1, 1)$ belongs
to all the 4-xels in $D \cup D^-$ and hence to all the xels in $Y$. Similarly, the vertex $(1,1,1,2)$ belongs to all the xels in $Z$. Moreover, $Y \cap Z = A(q,I' \cap D) \neq \emptyset$ because at least one element of $D \setminus \{q\}$ is in $I'$. Also, no xel in $Y \setminus Z = A(q,I' \cap D^c)$ intersects a xel in $Z \setminus Y = A(q,I' \cap D^r)$. Since $\bigcup X = \bigcup \text{Attach}(q,I')$, it follows from Proposition 11 that the three conditions of Theorem 2 hold with $I'$ in place of $I$, so $q$ is simple in $I'$ as required.

4.3. Proof of the “only if” parts of the main theorem

The “only if” part of assertion 1 is just Theorem 5. To prove the “only if” part of assertion 2, let $S$ be any MNS set of a 4D binary image $I$. Then $S$ is nonempty and is contained in some $2 \times 2 \times 2 \times 2$ block of 4-xels, by Theorem 5.

A set $T$ of 4-xels that is contained in some $2 \times 2 \times 2 \times 2$ block of 4-xels will be called a spanning set if there is no $2 \times 2 \times 2 \times 2$ block of 4-xels that contains $T$.

We now suppose that our MNS set $S$ is a minimal spanning set—i.e., we suppose $S$ is a spanning set but no proper subset of $S$ is a spanning set—and deduce that $S$ must be an 80-component of $I$. This will show that no minimal spanning set can be MNS without being a component, which (by Theorem 7) is enough to establish the “only if” part of assertion 2 of Theorem 8, since every spanning set contains a minimal spanning set.

For any two 4-xels $p$ and $q$ let $p - q$ denote the vector from the centroid of $q$ to the centroid of $p$. We define the $l_1$-diameter of $S$ to be $\max_{p,q \in S} \|p - q\|_1$, where $\|v\|_1$ is the $l_1$-norm of the vector $v$ (i.e., the sum of the absolute values of the four components of $v$). Since $S$ is a spanning set, the $l_1$-diameter of $S$ is at least 2, and is therefore equal to 2, 3, or 4.

4.3.1. Case 1: The $l_1$-diameter of $S$ is 4

In this case, since $S$ is a minimal spanning set, $S = \{q,a\}$ for some 4-xels $q$ and $a$ such that $\|q - a\|_1 = 4$. Note that $q \cap a$ consists of just one vertex, $v$ say. Let $\mathcal{P} = \bigcup \text{Attach}(q,I \setminus \{a\})$, so that $\bigcup \text{Attach}(q,I) = \mathcal{P} \cup \{v\}$. Since $S$ is MNS in $I$, it follows from Theorem 3 that $q$ is non-simple in $I \setminus \{a\}$ but $q$ is simple in $I$. The latter and Theorem 2 imply $\mathcal{P} \cup \{v\} = \bigcup \text{Attach}(q,I)$ is connected, and so either $v \in \mathcal{P}$ or $\mathcal{P} = \emptyset$. But $v \in \mathcal{P}$ would imply $\bigcup \text{Attach}(q,I) = \mathcal{P} \cup \{v\} = \mathcal{P} = \bigcup \text{Attach}(q,I \setminus \{a\})$, which (by Theorem 2) would make it impossible for $q$ to be simple in $I$ but non-simple in $I \setminus \{a\}$. Hence $\mathcal{P} = \emptyset$ and so, by Theorem 6, $S$ is an 80-component of $I$.

4.3.2. Case 2: The $l_1$-diameter of $S$ is 3

In this case it is readily confirmed that, since $S$ is a minimal spanning set, $S = \{q,a,b\}$ for some 4-xels $q$, $a$, and $b$ such that $\|q - a\|_1 = \|q - b\|_1 = 3$ and $\|a - b\|_1 = 2$. Let $q \cap a = e_a$ and $q \cap b = e_b$. Then $e_a$ and $e_b$ are edges and $e_a \cap e_b$ consists of just a vertex.

Let $\mathcal{P} = \bigcup \text{Attach}(q,I \setminus \{a,b\})$, so $\bigcup \text{Attach}(q,I) = \mathcal{P} \cup e_a \cup e_b$, $\bigcup \text{Attach}(q,I \setminus \{a\}) = \mathcal{P} \cup e_b$, and $\bigcup \text{Attach}(q,I \setminus \{b\}) = \mathcal{P} \cup e_a$. Since $S$ is MNS in $I$, it follows from Theorem 3 that $q$ is non-simple in $I \setminus \{a,b\}$ but $q$ is simple in $I$, in $I \setminus \{a\}$, and in $I \setminus \{b\}$. Since $q$ is simple in $I \setminus \{b\}$ but non-simple in $I \setminus \{a,b\}$, it follows from Theorem 2 and Proposition 10 that $\chi(\mathcal{P} \cap e_a) \neq 1$. 
As \( q \) is simple in \( I \), in \( I \setminus \{a\} \), and in \( I \setminus \{b\} \), and as \( \bigcup \text{Attach}(q, I) = P \cup e_a \cup e_b \), \( \bigcup \text{Attach}(q, I \setminus \{a\}) = P \cup e_b \), \( \bigcup \text{Attach}(q, I \setminus \{b\}) = P \cup e_a \), and \( \chi(e_a) = \chi(e_b) = \chi(e_a \cap e_b) = 1 \), it follows from Theorem 2 and the Inclusion-Exclusion Principle for Euler characteristics that:

\[
1 = \chi(P \cup e_a) = \chi(P) + 1 - \chi(P \cap e_a)
\]
\[
1 = \chi(P \cup e_b) = \chi(P) + 1 - \chi(P \cap e_b)
\]
\[
1 = \chi(P \cup e_a \cup e_b)
\]
\[
= \chi(P) + 1 + 1 - \chi(P \cap e_a) - \chi(P \cap e_b) - 1 + \chi(P \cap e_a \cap e_b)
\]
and therefore \( \chi(P) = \chi(P \cap e_a) = \chi(P \cap e_b) = \chi(P \cap e_a \cap e_b) \).

So, since \( \chi(P \cap e_a) \neq 1 \), \( \chi(P \cap e_a \cap e_b) \neq 1 \). Thus \( P \cap e_a \cap e_b = 0 \) (since \( e_a \cap e_b \) consists of just a vertex) and \( \chi(P \cap e_a \cap e_b) = 0 \). Therefore \( \chi(P) = \chi(P \cap e_a) = \chi(P \cap e_b) = 0 \), so \( P \cap e_a = 0 \). Now if \( P \neq 0 \) then \( P \cup e_a \) is disconnected, which contradicts Theorem 2 because \( P \cup e_a = \bigcup \text{Attach}(q, I \setminus \{b\}) \) and \( q \) is simple in \( I \setminus \{b\} \). Hence \( P = 0 \) and so, by Theorem 6, \( S \) is an \( 80 \)-component of \( I \).

### 4.3.3. Case 3: The \( I_1 \)-diameter of \( S \) is 2

In this case it is quite easy to verify that, since \( S \) is a minimal spanning set, \( S = \{q, a, b, c\} \) where \( ||x - y||_1 = 2 \) for all distinct \( x \) and \( y \) in \( S \). Let \( q \cap a = f_a \), \( q \cap b = f_b \), and \( q \cap c = f_c \). Then \( f_a, f_b, \) and \( f_c \) are 2-xels, every pair of them share an edge, and \( f_a \cap f_b \cap f_c \) consists of just a vertex.

Let \( P = \bigcup \text{Attach}(q, I \setminus \{a, b, c\}) \), so that \( \bigcup \text{Attach}(q, I \setminus \{a, b\}) = P \cup f_c \). Since \( S \) is MNS in \( I \), it follows from Theorem 3 that \( q \) is non-simple in \( I \setminus \{a, b, c\} \) but \( q \) is simple in \( I \setminus \{a, b\} \), \( I \setminus \{b, c\} \), and \( I \setminus \{a, b, c\} \). Since \( q \) is simple in \( I \setminus \{a, b\} \) but non-simple in \( I \setminus \{a, b, c\} \), it follows from Theorem 2 and Proposition 10 that \( \chi(P \cap f_c) \neq 1 \).

Since \( q \) is simple in \( I \setminus \{b, c\} \) and \( \bigcup \text{Attach}(q, I \setminus \{b, c\}) = P \cup f_a \), it follows from Theorem 2, the fact that \( \chi(x) = 1 \) for any xel \( x \), and the Inclusion–Exclusion Principle for Euler characteristics that \( 1 = \chi(P \cup f_a) = \chi(P) + 1 - \chi(P \cap f_a) \). Hence \( \chi(P) = \chi(P \cap f_a) \). By symmetrical arguments we must have:

\[
\chi(P) = \chi(P \cap f_a) = \chi(P \cap f_b) = \chi(P \cap f_c).
\]

Similarly, since \( q \) is simple in \( I \setminus \{c\} \) and \( \bigcup \text{Attach}(q, I \setminus \{c\}) = P \cup f_a \cup f_b \), we have

\[
1 = \chi(P \cup f_a \cup f_b) = \chi(P) + 1 + 1 - \chi(P \cap f_a) - \chi(P \cap f_b) - \chi(P \cap f_a \cap f_b)
\]
and so by Eq. (1) we have \( \chi(P) = \chi(P \cap f_a \cap f_b) \). By symmetrical arguments we must have:

\[
\chi(P) = \chi(P \cap f_a \cap f_b) = \chi(P \cap f_b \cap f_c) = \chi(P \cap f_a \cap f_c).
\]

Again, since \( q \) is simple in \( I \) and \( \bigcup \text{Attach}(q, I) = P \cup f_a \cup f_b \cup f_c \), we have

\[
1 = \chi(P \cup f_a \cup f_b \cup f_c) = \chi(P) + 1 + 1 - \chi(P \cap f_a) - \chi(P \cap f_b) - \chi(P \cap f_c) - \chi(P \cap f_a \cap f_b) - \chi(P \cap f_a \cap f_c) - \chi(P \cap f_b \cap f_c) - \chi(P \cap f_a \cap f_b \cap f_c)
\]
and so by Eqs. (1) and (2) we have:

\[
\chi(P) = \chi(P \cap f_a \cap f_b \cap f_c).
\]
Recalling that $\chi(P \cap f_z) \neq 1$, we see that Eqs. (1) and (3) imply $\chi(P \cap f_a \cap f_b \cap f_c) \neq 1$. Thus $P \cap f_a \cap f_b \cap f_c = \emptyset$ (since $f_a \cap f_b \cap f_c$ consists of just a vertex) and $\chi(P \cap f_a \cap f_b \cap f_c) = 0$. Therefore, by Eqs. (1)–(3), the 2-xel $f_a$ satisfies $\chi(P \cap f_a) = \chi(P \cap f_a \cap f_b) = 0$. Since $\chi(P \cap f_a) = 0$, $P \cap f_a$ either is empty or is the union of the four edges of $f_a$. But the latter is impossible because $f_a \cap f_b$ is an edge of $f_a$ that is not contained in $P$ (since $\chi(P \cap f_a \cap f_b) = 0$). Hence $P \cap f_a = \emptyset$. Now if $P \neq \emptyset$ then $P \cup f_a$ is disconnected, which contradicts Theorem 2 because $P \cup f_a = \bigcup \text{Attach}(q, I \setminus \{b, c\})$ and $q$ is simple in $I \setminus \{b, c\}$. So $P = \emptyset$ and, by Theorem 6, $S$ is an 80-component of $I$.

5. Concluding remarks

We have shown that a set of 4-xels can be MNS if and only if it is a nonempty subset of a $2 \times 2 \times 2 \times 2$ block of 4-xels, and can be MNS without being a component if and only if it is a nonempty subset of a $2 \times 2 \times 2$ block of 4-xels. This result provides the basis for a fairly general method, described in Section 2.4, of verifying that a proposed 4D thinning algorithm is topologically sound. Our proof of the result is based on a characterization of simple 4-xels that was given in [8], and the Inclusion–Exclusion Principle for Euler characteristics.

Our definition of simpleness and our interpretation TPres of the topology preservation requirement for thinning algorithms are consistent with the use of 80-adjacency to define connectedness on the 4-xels in a 4D binary image, and 8-adjacency to define connectedness on the 4-xels of the complement. But, instead of satisfying TPres, it is also possible for a thinning algorithm to preserve topology in a different sense that is consistent with the use of another pair of adjacency relations to define connectedness on 4-xels of the image and the complement. Just as in the 3D case one could use 18-adjacency on voxels in the image and 6-adjacency on the complement, or use 6-adjacency on voxels in the image and 26- or 18-adjacency on the complement, one might use 64- or 32-adjacency on the 4-xels in a 4D binary image and 8-adjacency on the complement, or 8-adjacency on the 4-xels in the image and 80-, 64-, or 32-adjacency on the complement. To deal with algorithms that preserve topology in these alternative senses, simpleness must be defined differently.

The authors are currently working on the problem of identifying all possible types of MNS sets in these cases.

References


