Minimal Non-Simple (MNS) Sets in 4-Dimensional Binary Images with (8,80)-Adjacency

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We present a new result that provides the basis for a fairly general method of determining if a given (parallel) local point-deletion operator for 4D binary images [e.g., a deletion operator used in a 4D parallel thinning algorithm] “preserves (8,80)-topology”.

8-adjacency on $\mathbb{Z}^4$ is analogous to 4-/6-adjacency on $\mathbb{Z}^2$ and $\mathbb{Z}^3$. 80-adjacency on $\mathbb{Z}^4$ is analogous to 8-/26-adjacency on $\mathbb{Z}^2$ and $\mathbb{Z}^3$.

For points $p = (p_1, p_2, p_3, p_4)$, $q = (q_1, q_2, q_3, q_4)$ in $\mathbb{Z}^4$,

- $p$ is 8-adjacent to $q$ iff $\|p - q\|_1 = \sum_{i=1}^{4} |p_i - q_i| = 1$
- $p$ is 80-adjacent to $q$ iff $\|p - q\|_{\infty} = \max_{1 \leq i \leq 4} |p_i - q_i| = 1$

“preserves (8,80)-topology” means preserves topology in a certain sense (which is defined in our paper) that is consistent with the use of 8-adjacency to define connectedness of sets of 1s and the use of 80-adjacency to define connectedness of sets of 0s in a 4D binary image.

Our Main Theorem is the “(8,80)-analog” of a similar (but simpler) theorem for (80,8)-topology preservation. [C.J. Gau and T.Y. Kong, Minimal nonsimple sets in 4D binary images, Graphical Models, 65:112–130, 2003.]

These results are analogs for $\mathbb{Z}^4$ of results established by Ronse (for $\mathbb{Z}^2$) and by Ma and Kong (for $\mathbb{Z}^3$).
An \((8,80)\)-**simple point** in a set \(Q \subseteq \mathbb{Z}^4\) is (roughly speaking) a point \(p \in Q\) which has the property that deletion of \(p\) from \(Q\) “preserves \((8,80)\)-topology”.

This is a 4D analog of the established 2D and 3D concepts of a \((4,8)\)-**simple** and a \((6,26)\)-**simple** point.

We will define \((8,80)\)-simple points later, but right now we define \((8,80)\)-simple sets in terms of the concept of an \((8,80)\)-simple **point**:

**Definition** Let \(Q \subseteq \mathbb{Z}^4\). Then \((8,80)\)-**simple** sets in \(Q\) are defined recursively, as follows:

- \(\emptyset\) is \((8,80)\)-simple in \(Q\).
- If \(M \subseteq Q\) and there is some point \(p \in M\) for which
  1. \(M \setminus \{p\}\) is \((8,80)\)-simple in \(Q\), and
  2. \(p\) is \((8,80)\)-simple in \(Q \setminus (M \setminus \{p\})\),
  then \(M\) is \((8,80)\)-simple in \(Q\).

Thus a nonempty set \(M \subseteq Q\) is \((8,80)\)-simple in \(Q\) if and only if the points of \(M\) can be arranged in a finite sequence \(p_0, p_1, \ldots, p_k\) in which

- \(p_0\) is \((8,80)\)-simple in \(Q\), and
- for \(1 \leq i \leq k\), \(p_i\) is \((8,80)\)-simple in \(Q \setminus \{p_0, p_1, \ldots, p_{i-1}\}\).

[Note the case \(k = 0\): If \(M\) is a singleton set \(\{p\}\), then \(M\) is \((8,80)\)-simple in \(Q\) **iff** \(p\) is \((8,80)\)-simple in \(Q\).]
If the set $D$ is (8,80)-simple in $Q$, then the deletion of $D$ from $Q$ “preserves topology” in our (8,80) sense.

*Given a proposed 4D parallel thinning operator $O$, how can we establish that the set of 1s that $O$ deletes from a binary image is always (8,80)-simple in the set of all 1s of that image?*

Let $D$ be the set of 1s that are deleted when the parallel thinning operator $O$ is applied to a 4D binary image whose set of 1s is $Q$. [Thus $D \subseteq Q \subseteq \mathbb{Z}^4$.]

Our Main Theorem will imply Theorem A below, which gives *local* conditions that are sufficient for the deletion of $D$ from $Q$ to be (8,80)-topology preserving.

**Definition:** A set $S \subseteq \mathbb{Z}^4$ will be called a *critical* set if there is an isometry of $\mathbb{Z}^4$ to itself that maps $S$ onto one of the 8 sets shown in the diagrams on the next page.

- Each critical set lies in a $2 \times 2 \times 2 \times 2$ block of 16 points.
- No critical set consists of more than 4 points.

**Theorem A:** Let $D$ be a finite subset of $Q \subseteq \mathbb{Z}^4$ and suppose that, for every critical set $M \subseteq D$, there is some point $p \in M$ such that $p$ is (8,80)-simple in $Q \setminus (M \setminus \{p\})$. Then $D$ is (8,80)-simple in $Q$. 
Critical Sets in $\mathbb{Z}^4$
To state our Main Theorem, we need (a 4D analog of) a concept introduced by Ronse in the mid-1980s:

**Definition** Let \( Q \subseteq \mathbb{Z}^4 \). A set \( M \subseteq Q \) is said to be \((8,80)\)-minimal-non-simple \([8,80]\)-MNS\] in \( Q \) if:

1. \( M \) is \((8,80)\)-non-simple in \( Q \), but
2. Every proper subset of \( M \) is \((8,80)\)-simple in \( Q \).

**Observation 1:** Every \((8,80)\)-non-simple finite subset of \( Q \) has a subset that is \((8,80)\)-MNS in \( Q \).

**Recall:** **Definition** Let \( Q \subseteq \mathbb{Z}^4 \). Then \((8,80)\)-simple sets in \( Q \) are defined recursively, as follows:

- \( \emptyset \) is \((8,80)\)-simple in \( Q \).
- If \( M \subseteq Q \) and there is some point \( p \in M \) for which
  1. \( M \setminus \{p\} \) is \((8,80)\)-simple in \( Q \), and
  2. \( p \) is \((8,80)\)-simple in \( Q \setminus (M \setminus \{p\}) \),
  then \( M \) is \((8,80)\)-simple in \( Q \).

From the above definitions we deduce:

**Observation 2:** If \( M \) is \((8,80)\)-MNS in \( Q \), then every point \( p \in M \) is \((8,80)\)-non-simple in \( Q \setminus (M \setminus \{p\}) \).

We say that a set \( M \subseteq \mathbb{Z}^4 \) can be \((8,80)\)-MNS if there exists a set \( Q \subseteq \mathbb{Z}^4 \) such that \( M \subseteq Q \) and \( M \) is \((8,80)\)-MNS in \( Q \).
Recall:  
We say that a set \( M \subseteq \mathbb{Z}^4 \) can be \((8,80)\)-MNS if there exists a set \( Q \subseteq \mathbb{Z}^4 \) such that \( M \subseteq Q \) and \( M \) is \((8,80)\)-MNS in \( Q \).

We can now state our

**MAIN THEOREM:** A set \( M \subseteq \mathbb{Z}^4 \) can be \((8,80)\)-MNS if and only if \( M \) is a critical set.

The Main Theorem implies our Theorem A!

**Theorem A:** Let \( D \) be a finite subset of \( Q \subseteq \mathbb{Z}^4 \) and suppose that, for every critical set \( M \subseteq D \), there is some point \( p \in M \) such that \( p \) is \((8,80)\)-simple in \( Q \setminus (M \setminus \{p\}) \). Then \( D \) is \((8,80)\)-simple in \( Q \).

Recall that, for all sets \( Q \subseteq \mathbb{Z}^4 \):

**Observation 1:** Every \((8,80)\)-non-simple finite subset of \( Q \) has a subset that is \((8,80)\)-MNS in \( Q \).

**Observation 2:** If \( M \) is \((8,80)\)-MNS in \( Q \), then every point \( p \in M \) is \((8,80)\)-non-simple in \( Q \setminus (M \setminus \{p\}) \).

**Proof of Thm. A:** By Observation 2, the hypotheses imply that no critical set \( M \subseteq D \) is \((8,80)\)-MNS in \( Q \).

This and the Main Theorem imply that no subset of \( D \) is \((8,80)\)-MNS in \( Q \), whence \( D \) is \((8,80)\)-simple in \( Q \) [by Observation 1]. //
Simple Points of Subsets of $\mathbb{Z}^n$

An $n$-xel in $\mathbb{R}^n$ is a closed $n$-D hypercube $X \subseteq \mathbb{R}^n$ such that: 1. Each edge of $X$ has length 1.
2. Each edge of $X$ is parallel to a coordinate axis.
3. The centroid of $X$ is a point in $\mathbb{Z}^n$.

Notation: For each point $p \in \mathbb{Z}^n$, we write $X(p)$ to denote the $n$-xel in $\mathbb{R}^n$ that is centered at $p$.

Definition: Let $p \in Q \subseteq \mathbb{Z}^n$. We say the point $p$ is $(3^n-1, 2n)$-simple in $Q$ if the polyhedron $\bigcup_{q \in Q} X(q)$ can be continuously deformed over itself (by deformation retraction) onto the polyhedron $\bigcup_{q \in Q \setminus \{p\}} X(q)$.

We say $p$ is $(2n, 3^n-1)$-simple in $Q$ if $p$ is $(3^n-1, 2n)$-simple in $\mathbb{Z}^n \setminus (Q \setminus \{p\}) \equiv (\mathbb{Z}^n \setminus Q) \cup \{p\}$.

When $n = 2$ or 3, these definitions of $(3^n-1, 2n)$- and $(2n, 3^n-1)$-simple points [i.e., $(8,4)$- and $(4,8)$-simple points when $n = 2$, and $(26,6)$- and $(6,26)$-simple points when $n = 3$] can be shown to be equivalent to the usual discrete definitions of such points (due to Rosenfeld and others) in the literature.

The advantage of the above definitions is that they apply in 4 dimensions, when $(2n, 3^n-1) = (8, 80)!$
For every $p \in \mathbb{Z}^n$, we define:

$$\text{Boundary}(X(p)) = \{ X(p) \cap X(q) \mid q \in \mathbb{Z}^n \setminus \{p\} \} \setminus \{ \emptyset \}$$

Each element of $\text{Boundary}(X(p))$ is called a **proper face** of $X(p)$.

We can use a **Schlegel diagram** to represent $\text{Boundary}(X(p))$ in $\mathbb{R}^{n-1} \cup \{\infty\}$. [$\infty$ denotes a “point at infinity” whose topological neighborhoods are the complements of the bounded subsets of $\mathbb{R}^{n-1}$.] Here are a 3-xel $X(p)$, and a Schlegel diagram of $\text{Boundary}(X(p))$ in $\mathbb{R}^2 \cup \{\infty\}$:

- One $(n-1)$-dimensional proper face of the $n$-xel $X(p)$ (in this example, the face EFGH) is represented by the closure of the “outside region” of the diagram.

- Otherwise, each $k$-dimensional proper face of $X(p)$ is represented in the Schlegel diagram by the $k$-cell that is the convex hull of the points which represent the vertices of that proper face.
A Schlegel diagram is a *topologically faithful* representation of $\text{Boundary}(X(p))$ in $\mathbb{R}^{n-1} \cup \{\infty\}$:

There is a homeomorphism

$$h : \bigcup \text{Boundary}(X(p)) \rightarrow \mathbb{R}^{n-1} \cup \{\infty\}$$

which maps each proper face of $X(p)$ onto the part of the Schlegel diagram that represents that proper face.

For $p \in \mathbb{Z}^4$, a Schlegel diagram of $\text{Boundary}(X(p))$ in $\mathbb{R}^3 \cup \{\infty\}$ looks like this:

For $p = (p_1, p_2, p_3, p_4)$:

$$++-+ = \text{the point that represents the vertex}$$

$$(p_1^{+\frac{1}{2}}, p_2^{-\frac{1}{2}}, p_3^{+\frac{1}{2}}, p_4^{+\frac{1}{2}})$$

of $X(p)$

$$+-++ = \text{the point that represents the vertex}$$

$$(p_1^{+\frac{1}{2}}, p_2^{-\frac{1}{2}}, p_3^{-\frac{1}{2}}, p_4^{+\frac{1}{2}})$$

of $X(p)$
**Definition:** Let \( p \in \mathbb{Z}^n \) and let \( Q \subseteq \mathbb{Z}^n \). Then the **attachment set** of \( p \) in \( Q \), denoted by \( \text{ATTACH}(p, Q) \), is the polyhedron \( \bigcup \{ X(p) \cap X(q) \mid q \in Q \setminus \{p\} \} \).

If \( p \notin Q \), and \( \bigcup_{q \in Q \cup \{p\}} X(q) \) is to be obtained by “gluing” \( X(p) \) onto \( \bigcup_{q \in Q} X(q) \), then \( \text{ATTACH}(p, Q) \) is “the subset of \( \bigcup \text{Boundary}(X(p)) \) on which we would apply glue”!

Let \( Q \) be the set of centers of the dark gray 2-xels.

If \( X(p) \) is any light gray 2-xel, then \( \text{ATTACH}(p, Q) \) is the part of \( \bigcup \text{Boundary}(X(p)) \) that is shown in black.

For all \( p, q \in \mathbb{Z}^n \) and all \( Q, Q_1, Q_2 \subseteq \mathbb{Z}^n \):
- \( \text{ATTACH}(p, Q \cup Q_2) = \text{ATTACH}(p, Q_1 \cup Q_2) \)
- \( \text{ATTACH}(p, \{p\}) = \emptyset \)
- \( \text{ATTACH}(p, \{q\}) = X(p) \cap X(q) \) if \( p \neq q \)
- \( \text{ATTACH}(p, Q \cup \{p\}) = \text{ATTACH}(p, Q) = \text{ATTACH}(p, Q \setminus \{p\}) \)
- \( \text{ATTACH}(p, \{q\}) = \emptyset \) iff \( q \) is not 80-adjacent to \( p \).
A 4D Example of How to Compute ATTACH(p, Q)

For brevity, when \( a, b, c, d \in \{0,1, \ldots, 9\} \) we write \( abcd \) for the point \((a, b, c, d)\), and write \( \text{abcd} \) for \( X(abcd) \).

[e.g., \( 3121 = X(3121) = \) the 4-xel centered at the point \((3,1,2,1)\).]

What is \( \text{ATTACH}(2222, \{3121, 2131, 1031, 1232\}) \)?

Write \(-\) for \( 2-\frac{1}{2} = \frac{3}{2} \). Write \( +\) for \( 2+\frac{1}{2} = \frac{5}{2} \).

Write \( \pm \) for the interval \([2-\frac{1}{2}, 2+\frac{1}{2}] = [\frac{3}{2}, \frac{5}{2}] \).

Faces of \( 2222 \) will be denoted by 4-strings of \(+\), \(-\), and \(\pm\):
E.g., \( -\pm+\pm = \{2-\frac{1}{2}\} \times [2-\frac{1}{2}, 2+\frac{1}{2}] \times \{2+\frac{1}{2}\} \times [2-\frac{1}{2}, 2+\frac{1}{2}] \)

\[
\text{ATTACH}(2222, \{3121, 2131, 1031, 1232\}) = \bigcup \{ 2222 \cap 3121, \ 2222 \cap 2131, \ 2222 \cap 1031, \ 2222 \cap 1232 \} = \bigcup \{ \pm-\pm-, \ \pm+-+, \ \emptyset, \ -\pm\pm \}
\]

\( \text{ATTACH}(2222, \{3121,2131,1031,1232\}) \) in a Schlegel Diagram

of Boundary(2222)

[= Boundary(X(2,2,2,2))]
A Discrete Characterization of 4D Simple Points

Theorem 1  Let $p \in Q \subseteq \mathbb{Z}^4$. Then $p$ is $(8,80)$-simple in $Q$ iff the following all hold:
(a) The polyhedron $\text{ATTACH}(p, \mathbb{Z}^4 \setminus Q)$ is connected.
(b) $\bigcup \text{Boundary}(p) \setminus \text{ATTACH}(p, \mathbb{Z}^4 \setminus Q)$ is connected.
(c) $\chi(\text{ATTACH}(p, \mathbb{Z}^4 \setminus Q)) = 1$.

$\chi(\cdot)$ denotes the Euler number: If $A = \text{ATTACH}(p, \mathbb{Z}^4 \setminus Q)$, then

$$\chi(A) = \sum_{i=0}^{3} (-1)^i c_i(A) = c_0(A) - c_1(A) + c_2(A) - c_3(A)$$

where $c_i(A)$ = the number of $i$-dimensional proper faces of $X(p)$ that lie in $A = \text{ATTACH}(p, \mathbb{Z}^4 \setminus Q)$.

Note: In our paper, the set of all proper faces of $X(p)$ that lie in $\text{ATTACH}(p, \mathbb{Z}^4 \setminus Q)$ is denoted by $\text{Coattach}(p, Q)$.

It follows from our definition of $(8,80)$-simple points

Recall: We say $p$ is $(2n, 3^n-1)$-simple in $Q$ if $p$ is $(3^n-1, 2n)$-simple in $\mathbb{Z}^n \setminus (Q \setminus \{p\})$.

that Theorem 1 is equivalent to:

Theorem 2 (Kong, 1997; Gau and Kong, 2002) Let $p \in Q \subseteq \mathbb{Z}^4$. Then $p$ is $(80,8)$-simple in $Q$ iff the following all hold:
(a) The polyhedron $\text{ATTACH}(p, Q \setminus \{p\})$ is connected.
(b) The set $\bigcup \text{Boundary}(p) \setminus \text{ATTACH}(p, Q \setminus \{p\})$ is connected.
(c) $\chi(\text{ATTACH}(p, Q \setminus \{p\})) = 1$. 
Our proof of the Main Theorem

Recall: MAIN THEOREM: A set \( M \subseteq \mathbb{Z}^4 \) can be (8,80)-MNS if and only if \( M \) is a critical set.

is based on:

A. The characterization of (8,80)-simple points \( p \) in terms of \( \text{ATTACH}(p, \mathbb{Z}^4 \setminus Q) \) [i.e., Thm. 1 above].

B. The following characterization of (8,80)-MNS sets (which is a special case of a more general result obtained by Kong and Ma in 1993):

Theorem 3: Let \( M \subseteq Q \subseteq \mathbb{Z}^4 \). Then \( M \) is (8,80)-MNS in \( Q \) iff \( M \neq \emptyset \) and the following hold for every \( p \in M \):

1. \( p \) is (8,80)-non-simple in \( (Q \setminus M) \cup \{p\} \), but
2. \( p \) is (8,80)-simple in \( (Q \setminus M) \cup S \) whenever \( \{p\} \subsetneq S \subseteq M \).

[Assertion 1 is just our earlier Observation 2!]

Theorem 3 implies:

Cor. 1: Let \( M \subseteq \mathbb{Z}^4 \) and suppose \( M \) can be (8,80)-MNS. Then, for every \( X \subseteq M \), \( M \setminus X \) can be (8,80)-MNS.

Proof: Thm. 3 implies that if \( M \) is (8,80)-MNS in \( Q \) then \( M \setminus X \) is (8,80)-MNS in \( Q \setminus X \). //
Proof of the “If” Part of the Main Theorem

Recall:

Cor. 1: Let $M \subseteq \mathbb{Z}^4$ and suppose $M$ can be $(8,80)$-MNS. Then, for every $X \subseteq M$, $M \setminus X$ can be $(8,80)$-MNS.

MAIN THEOREM: A set $M \subseteq \mathbb{Z}^4$ can be $(8,80)$-MNS if and only if $M$ is a critical set.

In view of Cor. 1, the “if” part of the Main Theorem only needs to be verified for critical sets of types 2, 5, 7, and 8 [because each of the other four types of critical set is a subset of a set of one of these types].

1

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Proof of the “If” Part of the Main Thm. (contd.)

It remains to show that critical sets of types 2, 5, 7, and 8 can be (8,80)-MNS.

Recall:

**Theorem 3:** Let $M \subseteq Q \subseteq \mathbb{Z}^4$. Then $M$ is (8,80)-MNS in $Q$ iff $M \neq \emptyset$ and the following hold for every $p \in M$:

1. $p$ is (8,80)-non-simple in $(Q \setminus M) \cup \{p\}$, but
2. $p$ is (8,80)-simple in $(Q \setminus M) \cup S$ whenever $\{p\} \subset S \subseteq M$.

**Case 1: $M$ is a Set of Type 2**

WLOG, $M = \{1\} \times \{1\} \times \{4,5\} \times \{1\}$. Theorem 3 tells us $M$ is (8,80)-MNS in $Q = \{0,1,2\} \times \{1\} \times \{4,5\} \times \{1\}$.

**Cases 2, 3, and 4: $M$ is a Set of Type 5, 7, or 8**

Let $Q = \text{the } 2 \times 2 \times 2 \times 2 \text{ block of 16 points that contains } M$.

In each of the three cases, Theorem 3 tells us that $M$ is (8,80)-MNS in $Q$. 
Outline of the Proof of the “Only If” Part
[i.e., proof that only critical sets can be (8,80)-MNS]

**Observation 3:** If \( p \in T \subseteq \mathbb{Z}^4 \), then the answer to the question “Is \( p \) (8,80)-simple in \( T \)” depends only on \( \text{ATTACH}(p, \mathbb{Z}^4 \setminus T) \).

This follows from:

**Theorem 1** Let \( p \in Q \subseteq \mathbb{Z}^4 \). Then \( p \) is (8,80)-simple in \( Q \) iff the following all hold:

(a) The polyhedron \( \text{ATTACH}(p, \mathbb{Z}^4 \setminus Q) \) is connected.

(b) The set \( \bigcup \text{Boundary}(p) \setminus \text{ATTACH}(p, \mathbb{Z}^4 \setminus Q) \) is connected.

(c) \( \chi(\text{ATTACH}(p, \mathbb{Z}^4 \setminus Q)) = 1 \).

Now recall:

**Theorem 3:** Let \( M \subseteq Q \subseteq \mathbb{Z}^4 \). Then \( M \) is (8,80)-MNS in \( Q \) iff \( M \neq \emptyset \) and the following hold for every \( p \in M \):

1. \( p \) is (8,80)-non-simple in \( (Q \setminus M) \cup \{p\} \), but
2. \( p \) is (8,80)-simple in \( (Q \setminus M) \cup S \) whenever \( \{p\} \subsetneq S \subseteq M \).

Theorem 3 and the above observation imply:

If a set \( M \) is (8,80)-MNS in a set \( Q \) then, for every \( p \in M \),

\[
\text{ATTACH}(p, \mathbb{Z}^4 \setminus ((Q \setminus M) \cup \{p\})) \\
\neq \text{ATTACH}(p, \mathbb{Z}^4 \setminus ((Q \setminus M) \cup S))
\]

whenever \( \{p\} \subsetneq S \subseteq M \). \hspace{1cm} (★)
We have just shown:
If a set $M$ is $(8,80)$-MNS in a set $Q$ then, for every $p \in M$,
$$\text{ATTACH}(p, \mathbb{Z}^4 \backslash ((Q \setminus M) \cup \{p\})) \neq \text{ATTACH}(p, \mathbb{Z}^4 \backslash ((Q \setminus M) \cup S))$$
whenever $\{p\} \subsetneq S \subseteq M$. \hfill (★)

Putting $S = \{p, r\}$ in (★), where $r \in M$ and $r \neq p$, we get:

If a set $M$ is $(8,80)$-MNS in a set $Q$ then, for all distinct $p, r \in M$,
$$\text{ATTACH}(p, \mathbb{Z}^4 \backslash ((Q \setminus M) \cup \{p\})) \neq \text{ATTACH}(p, \mathbb{Z}^4 \backslash ((Q \setminus M)\cup\{p, r\}))$$
\hfill (∗)

Also recall that, for all $p \in \mathbb{Z}^4$ and all $Q_1, Q_2 \subseteq \mathbb{Z}^4$:
$$\text{ATTACH}(p, Q_1 \cup Q_2) = \text{ATTACH}(p, Q_1) \cup \text{ATTACH}(p, Q_2)$$

Cor. 2: Let $M$ be a subset of $\mathbb{Z}^4$ that is not contained in a $2 \times 2 \times 2 \times 2$ block. Then $M$ cannot be $(8,80)$-MNS.

Proof. Suppose $M$ were $(8,80)$-MNS in a set $Q$. As $M$ is not contained in a $2 \times 2 \times 2 \times 2$ block, there exist $p, r \in M$ such that $r$ does not lie in the 80-neighborhood of $p$.

Now $\text{ATTACH}(p, \{r\}) = \emptyset$ and therefore

[since $\mathbb{Z}^4 \backslash ((Q \setminus M) \cup \{p\}) = \{r\} \cup \mathbb{Z}^4 \backslash ((Q \setminus M) \cup \{p, r\})$]
$$\text{ATTACH}(p, \mathbb{Z}^4 \backslash ((Q \setminus M) \cup \{p\})) = \text{ATTACH}(p, \mathbb{Z}^4 \backslash ((Q \setminus M) \cup \{p, r\}))$$

This contradicts (∗)! //
Recall: If a set $M$ is $(8,80)$-MNS in a set $Q$ then, for all distinct $p, r \in M$, 
\[ \text{ATTACH}(p, \mathbb{Z}^4 \setminus ((Q \setminus M) \cup \{p\})) \neq \text{ATTACH}(p, \mathbb{Z}^4 \setminus ((Q \setminus M) \cup \{p, r\})) \]  
\[ \tag{\star} \]

Cor. 3: Let $M \subseteq \mathbb{Z}^4$ be a set that strictly contains a set of type 2 or type 5. Then $M$ cannot be $(8,80)$-MNS.

Proof. Suppose $M$ is $(8,80)$-MNS in a set $Q$. Then $M$ is contained in a $2 \times 2 \times 2 \times 2$ block, by Cor. 2.

CASE I: $M$ strictly contains a set $S$ of type 2.

Let $r \in M \setminus S$, and let $S = \{p, q\}$, where distance($r, q$) < distance($r, p$).

Case II: $M$ strictly contains a set $S$ of type 5.

Let $S = \{p, r\}$ and let $q \in M \setminus S$.

In both cases, \[ \text{ATTACH}(p, \{q\}) \supseteq \text{ATTACH}(p, \{r\}) \]  
\[ \tag{\ddagger} \]

Since \[ \mathbb{Z}^4 \setminus ((Q \setminus M) \cup \{p\}) = (\mathbb{Z}^4 \setminus ((Q \setminus M) \cup \{p, r\})) \cup \{r\}, \]
\[ \text{ATTACH}(p, \mathbb{Z}^4 \setminus ((Q \setminus M) \cup \{p\})) \]
\[ = \text{ATTACH}(p, \mathbb{Z}^4 \setminus ((Q \setminus M) \cup \{p, r\})) \cup \text{ATTACH}(p, \{r\}) \]  
\[ \tag{*} \]

But $q \in M$, so $q \in \mathbb{Z}^4 \setminus ((Q \setminus M) \cup \{p, r\})$, and hence \[ \text{ATTACH}(p, \mathbb{Z}^4 \setminus ((Q \setminus M) \cup \{p, r\})) \]
\[ \supseteq \text{ATTACH}(p, \{q\}) \supseteq \text{ATTACH}(p, \{r\}) \] [by (\ddagger)]

This and (\star) imply that the two sides of (\star) are equal —contradiction! //
Recall:
**Cor. 1:** Let $M \subseteq \mathbb{Z}^4$ and suppose $M$ can be $(8,80)$-MNS. Then, for every $X \subseteq M$, $M \setminus X$ can be $(8,80)$-MNS.

**Cor. 2:** Let $M$ be a subset of $\mathbb{Z}^4$ that is not contained in a $2 \times 2 \times 2 \times 2$ block. Then $M$ cannot be $(8,80)$-MNS.

**Cor. 3:** Let $M \subseteq \mathbb{Z}^4$ be a set that strictly contains a set of type 2 or type 5. Then $M$ cannot be $(8,80)$-MNS.

To complete the proof of the “only if” part of the Main Theorem (i.e., to show that a set $M$ can be $(8,80)$-MNS only if it is a set of one of the types 1–8) it is enough to establish the following two claims:

**Claim I:** Every set $M \subseteq \mathbb{Z}^4$ that is not a set of one of the types 1 – 8 satisfies at least one of the following conditions:

(i) $M$ is not contained in a $2 \times 2 \times 2 \times 2$ block.
(ii) $M$ strictly contains a set of type 2 or type 5.
(iii) $M$ has a subset that is congruent to one of the following 3 sets:

![Diagram](image)

**Claim II:** None of the 3 sets specified in condition (iii) of Claim I can be $(8,80)$-MNS.
How the Proof of the Main Theorem is Completed

Claim I: Every set \( M \subseteq \mathbb{Z}^4 \) that is not a set of one of the types 1 – 8 satisfies at least one of the following conditions:
(i) \( M \) is not contained in a \( 2 \times 2 \times 2 \times 2 \) block.
(ii) \( M \) strictly contains a set of type 2 or type 5.
(iii) \( M \) has a subset that is congruent to one of the following 3 sets:

Claim II: None of the 3 sets specified in condition (iii) of Claim I can be \((8,80)\)-MNS.

Claim I is established by case-checking.

To prove Claim II, we note that the 4 points of each of the 3 sets can be named \( p, q_1, q_2, q_3 \) in such a way that:
\[
X(p) \cap X(q_1) \cap X(q_2) = X(p) \cap X(q_1) \cap X(q_2) \cap X(q_3) \neq \emptyset \quad (♠)
\]

Thm. 3 tells us that if \( M \) is \((8,80)\)-MNS in a set \( Q \), then
(1) \( p \) is \((8,80)\)-non-simple in \((Q \setminus M) \cup \{p\}\), but
(2) \( p \) is \((8,80)\)-simple in \((Q \setminus M) \cup S \) whenever \( \{p\} \subset S \subseteq M \)

Using the characterization of \((8,80)\)-simple points given by Theorem 1, as well as the Inclusion-Exclusion Principle for Euler numbers, we deduce that when \( M = \{p, q_1, q_2, q_3\} \) conditions (1) and (2) are inconsistent with \((♠)\).   //
Concluding Remarks

MAIN THEOREM: A set $M \subseteq \mathbb{Z}^4$ can be (8,80)-minimal-non-simple iff $M$ is a critical set — i.e., iff $M$ is a set of one of the following 8 types:

The following corollary of the Main Theorem is useful for showing that a 4D (parallel) deletion operator $O$ is (8,80)-topology-preserving. [To apply it, let $D$ be the set of 1s that $O$ deletes from an image whose set of 1s is $Q$.]

Theorem A: Let $D$ be a finite subset of $Q \subseteq \mathbb{Z}^4$ and suppose that, for every critical set $M \subseteq D$, there is some $p \in M$ such that $p$ is (8,80)-simple in $Q \setminus (M \setminus \{p\})$. Then $D$ is (8,80)-simple in $Q$. 