Exercises Relating to Programming Assignment 2

Consider the problem of finding an optimal parenthesization of a matrix chain product \(A_1A_2A_3A_4A_5A_6\) in which each \(A_i\) is a \(p_{i-1} \times p_i\) matrix and the sequence of dimensions \((p_0, p_1, p_2, p_3, p_4, p_5, p_6)\) is \((30, 35, 15, 5, 10, 20, 25)\).

As in Sec. 15.2, for \(1 \leq i < j \leq 6\) we write \(m[i, j]\) to denote the minimum number of scalar multiplications needed to compute the product of the \(j - i + 1\) matrices \(A_i \ldots A_j\) and we write \(s[i, j]\) to denote an integer \(k\) between \(i\) and \(j - 1\) (inclusive) for which \(m[i, k] + m[k+1, j] + p_i \cdot p_k \cdot p_j\) is minimal—see (15.7) on p. 374.

The questions below test your understanding of how the procedure MATRIX-CHAIN-ORDER (p. 375) computes values of \(m[i, j]\) and \(s[i, j]\), and how the computed values \(s[i, j]\) are used by the procedure PRINT-OPTIMAL-PARENS (p. 377) to print an optimal parenthesization. When answering the questions, assume that \(m[i, j]\) and \(s[i, j]\) have already been computed for \(j \leq i + 2\) (\(1 \leq i < j \leq 6\)) and also for \([i, j] = [1, 4]\), and assume that the values are as follows:

1: \[m[1, 3] = 7875 \quad m[2, 4] = 4375 \quad m[3, 5] = 2500 \quad m[4, 6] = 3500\]
2: \[m[1, 2] = 15750 \quad m[2, 3] = 2625 \quad m[3, 4] = 750 \quad m[4, 5] = 1000 \quad m[5, 6] = 5000\]
3: \[s[1, 4] = 3\]
4: \[s[1, 3] = 1 \quad s[2, 4] = 3 \quad s[3, 5] = 3 \quad s[4, 6] = 5\]
5: \[s[1, 2] = 1 \quad s[2, 3] = 2 \quad s[3, 4] = 3 \quad s[4, 5] = 4 \quad s[5, 6] = 5\]

The values shown on the rows numbered 1, 2, and 3 above would be computed at the 1st, 2nd, and 3rd iterations of the for \(l = 2\) to \(n\) loop that begins on line 5 of MATRIX-CHAIN-ORDER. Questions 1 – 9 below require you to calculate other \(m[i, j]\) and \(s[i, j]\) values that would be computed at the 3rd, 4th, or 5th iterations of the same loop. Assume the above values are correct when answering the questions.

1. [1 pt.] Which of the following is the value of \(m[2, 5]\)? Circle the answer:
   (a) 6135   (b) 7125   (c) 7770   (d) 7980   (e) 8140

2. [1 pt.] Which of the following is the value of \(m[3, 6]\)? Circle the answer:
   (a) 5375   (b) 6540   (c) 7770   (d) 7980   (e) 8240

3. [1 pt.] Which one of the following gives correct values of \(s[2, 5]\) and \(s[3, 6]\)? Circle the answer:
   (a) \(s[2, 5] = 2; s[3, 6] = 3\)   (b) \(s[2, 5] = 2; s[3, 6] = 4\)   (c) \(s[2, 5] = 3; s[3, 6] = 3\)
   (d) \(s[2, 5] = 3; s[3, 6] = 5\)   (e) \(s[2, 5] = 4; s[3, 6] = 4\)

4. [1 pt.] Which of the following is the value of \(m[1, 5]\)? Circle the answer:
   (a) 9356   (b) 10010   (c) 11770   (d) 11875   (e) 13345

5. [1 pt.] Which of the following is the value of \(m[2, 6]\)? Circle the answer:
   (a) 10135   (b) 10440   (c) 10500   (d) 11985   (e) 12140

6. [1 pt.] Which one of the following gives correct values of \(s[1, 5]\) and \(s[2, 6]\)? Circle the answer:
   (a) \(s[1, 5] = 1; s[2, 6] = 3\)   (b) \(s[1, 5] = 3; s[2, 6] = 3\)   (c) \(s[1, 5] = 3; s[2, 6] = 4\)
   (d) \(s[1, 5] = 3; s[2, 6] = 5\)   (e) \(s[1, 5] = 4; s[2, 6] = 5\)

7. [1 pt.] Which of the following is the value of \(m[1, 6]\)? Circle the answer:
   (a) 15125   (b) 15400   (c) 15770   (d) 15980   (e) 16210

8. [1 pt.] Which of the following is the value of \(s[1, 6]\)? Circle the answer:
   (a) 1   (b) 2   (c) 3   (d) 4   (e) 5

9. [1 pt.] Write down an optimal parenthesization of the product \(A_1A_2A_3A_4A_5A_6\).

If you need more practice, do Exercise 15.2-1 (p. 378). That exercise requires much more calculation than is needed to answer questions 1 – 9 above, as no values of \(m[i, j]\) and \(s[i, j]\) are provided to you.

Answers to questions 1 – 9 and to Exercise 15.2-1 are given on the next page.
Answers:

The right choice for question 1 is (b): $m[2,5]$ can be calculated using (15.7) on p. 374. The calculation is described in detail in the caption of Fig. 15.5 on p. 376: The value of $k \in \{2,3,4\}$ that minimizes $m[2, k] + m[k+1, 5] + p_1 p_k p_5$ is $k = 3$, and the minimal value is 7125. So $s[2,5] = 3$ and $m[2,5] = 7125$.

We deduce from this and similar calculations based on (15.7) that the right choices for questions 2 – 8 are 2(a), 3(c), 4(d), 5(c), 6(b), 7(a), and 8(c); Fig. 15.5 confirms this.

Question 9: As $s[1,6] = 3$ (from the answer to q. 8) and you are told that $s[1,3] = 1$ and $s[4,6] = 5$, an optimal parenthesization is $((A_1(A_2A_3))(A_4A_5)A_6))$. The last sentence on p. 377 confirms this.

Exercise 15.2-1: Using (15.7) with $\langle p_0, p_1, p_2, p_3, p_4, p_5, p_6 \rangle = (5, 10, 3, 12, 5, 50, 6)$, we can deduce the following:

- $s[1,2] = 1, m[1,2] = 150$
- $s[2,3] = 2, m[2,3] = 360$
- $s[3,4] = 3, m[3,4] = 180$
- $s[4,5] = 4, m[4,5] = 3000$
- $s[5,6] = 5, m[5,6] = 1500$

- $s[1,3] = 2, m[1,3] = 330$
- $s[2,4] = 2, m[2,4] = 330$
- $s[3,5] = 4, m[3,5] = 930$
- $s[4,6] = 4, m[4,6] = 1860$

- $s[1,4] = 2, m[1,4] = 405$
- $s[2,5] = 2, m[2,5] = 2430$
- $s[3,6] = 4, m[3,6] = 1770$

- $s[1,5] = 4, m[1,5] = 1655$

- $s[1,6] = 2, m[1,6] = 2010$

Since $s[1,6] = 2$ and $s[3,6] = 4$, an optimal parenthesization is: $((A_1 A_2)((A_3 A_4)(A_5 A_6))$