**Homework 2 [Not for Credit]**

**A. p. 74: 4.1-1, 4.1.2*, 4.1-4**

*Do 4.1.2 without looking at the solution that was presented in class!

Read ex. 4.1-5 on p. 75 before you do the following related exercise:


(i) Let $x$ be the sum of $A[9 .. 17]$. Write an expression whose value is the sum of the maximum subarray of the form $A[i .. 18]$. Your expression may involve $x$ and $A[18]$, and may use the $\ldots ? \ldots : \ldots$ operator of C++ / Java or the max function.

(ii) Let $y$ be the sum of $A[3 .. 6]$ and let $z$ be the value of an expression that is a solution to (i). Write an expression whose value is the sum of the maximum subarray of $A[1 .. 18]$. Your expression may involve $y$ and $z$, and may use the $\ldots ? \ldots : \ldots$ operator of C++ / Java or the max function.

**B. pp. 92-3: 4.4-1, 4.4-2, 4.4-3*, 4.4-4, 4.4-6, 4.4-7, 4.4-8**

Most of these problems ask you to use the "substitution method" to verify your answer. However, you need not do the verification parts of those problems until after the substitution method (see Sec. 4.3) has been covered in class.

*In 4.4-3, assume arguments $n < 5$ are base cases—i.e., the recurrence only applies when $n \geq 5$.

In your solutions you may use the following 3 facts without proving them:

(These facts are easily proved. For example, on putting $k = l$ in the hypothesis of (3)(i) and $k = l - 1$ in that of (3)(ii) we deduce $G > \frac{n}{d^l} - C$ and $G \leq \frac{n}{d^{l-1}} + C$ respectively, from which (3)(i) and (3)(ii) follow.)

1. The sum of an $n$-term arithmetic progression $a + (a+d) + (a+2d) + \ldots + (a+(n-1)d)$, where $a$ and $d$ are constants, is $na + n(n-1)d/2$; if $d > 0$, this is asymptotically positive and is $\Theta(n^2) = \Theta(\text{number of terms})$.
2. The sum of an $n$-term geometric progression $1 + r + \ldots + r^{n-1}$, where $r$ is a constant and $r \neq 1$, is $(r^n - 1)/(r-1)$; this is $\Theta(r^n) = \Theta(\text{number of terms})$ if $r > 1$ but is $\Theta(1)$ if $|r| < 1$.
3. Say that a value $a$ is a base case just if $a < G$ (where $G$ is some positive constant). Let $n = a_1 > a_2 > \ldots > a_l$ be a sequence whose last term $a_l$ is a base case value but whose $2^{nd}$-last term $a_{l-1}$ is not a base case value. Then:
   - (i) If each term $a_l$ satisfies $a_l \geq n/d - C$ for some positive constants $d > 1$ and $C < G$, the length $l$ of the sequence must be $\geq \log d n - O(1)$.
   - (ii) If each term $a_k$ satisfies $a_k \leq n/d + C$ for some positive constants $d > 1$ and $C < G$, the length $l$ of the sequence must be $\leq \log d n + O(1)$.

**Example 1:** In a recursion tree of the merge-sort recurrence $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \Theta(n)$ whose root represents the argument value $n$, a node at level $k$ represents an argument value that lies between $\lceil n/2^k \rceil$ and $\lfloor n/2^k \rfloor$ and therefore lies between $n/2^k$ and $n/2^k + 1$. On applying (3) with $d = 2$ and $C = 1$ (to the argument values represented by a chain of nodes in which the first node is the root, each node is the parent of the next, and the last node is a leaf) we deduce that each leaf has level $\log_2 n \pm O(1)$. This implies there are at least $\log_2 n - O(1)$ lower levels which consist entirely of nodes that are not leaves, and also implies that no more than $O(1)$ levels of the tree contain leaves.

**Example 2:** In a recursion tree of the recurrence $T(n) = 4T(n/2 + 2) + n$ (see exercise 4.4-3) whose root represents the argument value $n$, it is readily confirmed (e.g., by induction on $k$) that the argument value represented by any node at level $k$ lies between $n/2^k$ and $n/2^k + 4$. On applying (3) with $d = 2$ and $C = 4$, we see that each leaf of the tree has level $\log_2 n \pm O(1)$.

**Example 3:** For a recursion tree of the recurrence $T(n) = T(n/3) + T(2n/3) + cn$ (see exercise 4.4-6), when we apply (3)(i) with $d = 3$ and (3)(ii) with $d = 3/2$ we see that each leaf in the tree has level $\geq \log_2 n - O(1)$ but has level $\leq \log_{\sqrt{3}} n + O(1)$, where $n$ is the argument value represented by the root.

**C. pp. 82-3: 4.2-1†, 4.2-3, 4.2-4, 4.2-5, 4.2-6, 4.2-7**

†For 4.2-1: “Show your work” means you should at least write down the values of each of the seven products $P_1, P_2, \ldots, P_7$, and also write down the addition and/or subtraction expression (involving two or four of those seven products) that yields each of the four elements of the matrix product.