Some Properties of Graphs That Have Hamilton Circuits or Hamilton Paths

Properties (a) – (j) below can be used to show that a graph has no Hamilton circuit / path of a certain kind, or that it has no Hamilton circuit / path at all. Some of these properties can also be used to reduce the search space when we are trying to find a Hamilton circuit or path.

(a) If a graph has a Hamilton circuit $C$, then each vertex of the graph is incident with exactly 2 edges of the circuit $C$.

(b) A Hamilton circuit cannot contain all the edges of a shorter simple circuit.

(c) If a graph has a Hamilton path $P$ from a vertex $u$ to a vertex $v$, then each of $u$ and $v$ is incident with exactly 1 edge of the path $P$, and each of the other vertices of the graph is incident with exactly 2 edges of the path $P$.

(d) A Hamilton path cannot contain all the edges of a simple circuit.

(e) A Hamilton path from a vertex $u$ to a vertex $v$ cannot contain all the edges of a shorter path from $u$ to $v$.

(f) Let $G$ be a graph that has a Hamilton circuit, and let $k$ be any integer such that $0 < k < |V(G)|$. Then, when we remove $k$ vertices from $G$, the resulting graph will have at most $k$ connected components.

(g) Let $G$ be a graph that has a Hamilton path, and let $k$ be any integer such that $0 < k < |V(G)|$. Then:
   1. If we remove $k$ vertices from $G$, the resulting graph will have at most $k + 1$ components.
   2. If we remove $k$ vertices from $G$ that include an endpoint of a Hamilton path of $G$, the resulting graph will have at most $k$ components.
   3. If we remove $k$ vertices from $G$ that include both endpoints of a Hamilton path of $G$, the resulting graph will have at most $k – 1$ components.

In properties (h) – (j), $G$ denotes a bipartite graph that has a bipartition $(R, B)$, and a vertex of $G$ is said to be red or blue according to whether it belongs to $R$ or to $B$.

(h) If $G$ has a Hamilton circuit, then $G$ has just as many red vertices as blue vertices, and so $|V(G)|$ is even.

(i) If $G$ has a Hamilton path, then the number of red vertices of $G$ and the number of blue vertices of $G$ are equal or differ by just one.

(j) If $G$ has just as many red vertices as blue vertices, then the first and last vertices of any Hamilton path of $G$ have opposite colors. Otherwise, the first and last vertices of any Hamilton path of $G$ are both red or both blue according to whether $G$ has more red or more blue vertices.