Some Properties of Graphs That Have Hamilton Circuits or Hamilton Paths

Properties (a) – (j) below are equivalent to the properties stated in the following document.

Let $G$ be any undirected graph. As usual, $V(G)$ denotes the vertex set of $G$.

Suppose $G$ has a Hamilton circuit $C$. Then:

(a) For all $v \in V(G)$, exactly 2 edges of $C$ are incident with $v$.
(b) $C$ cannot contain all the edges of a shorter simple circuit of $G$.

Suppose $G$ has a Hamilton path $P$ that begins at $a \in V(G)$ and ends at $b \in V(G)$. Then:

(c) For $v \in \{a, b\}$, exactly 1 edge of $P$ is incident with $v$;
   for all $v \in V(G) - \{a, b\}$, exactly 2 edges of $P$ are incident with $v$.
(d) $P$ cannot contain all the edges of a simple circuit of $G$.
(e) $P$ cannot contain all the edges of a shorter path from $a$ to $b$.

Let $S$ be any nonempty proper subset of $V(G)$. Then:

(f) $G - S$ has $\leq |S|$ components if $G$ has a Hamilton circuit.
(g1) $G - S$ has $\leq |S| + 1$ components if $G$ has a Hamilton path.
(g2) $G - S$ has $\leq |S|$ components if $G$ has a Hamilton path that begins or ends in $S$.
(g3) $G - S$ has $\leq |S| - 1$ components if $G$ has a Hamilton path that begins and ends in $S$.

Now suppose the graph $G$ is bipartite and has a bipartition $(R, B)$. Then:

(h) $|R| = |B|$ (and so $|V(G)|$ must be even) if $G$ has a Hamilton circuit.
(i) $|R| - |B| = -1$, 0, or +1 if $G$ has a Hamilton path.
(j) If $|R| \neq |B|$, then the first and the last vertices of any Hamilton path of $G$ must both belong to the larger of the two vertex classes $R$ and $B$. If $|R| = |B|$, then the first and the last vertices of any Hamilton path of $G$ must belong to different vertex classes.